MAT 102/EXL
EXL COMPONENT #13
RADICALS II

Rational Exponents and Laws of Radicals

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$$

$$a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}}$$

$$\sqrt[n]{a^n} = (\sqrt[n]{a})^n = a$$

$$\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$$

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$$

Rationalizing Denominators and Numerators

Rationalize each denominator.

a) $\sqrt{\frac{4}{5b}}$ **b)** $\frac{\sqrt[3]{a}}{\sqrt[3]{9x}}$ **c)** $\frac{3x}{\sqrt[5]{2x^2y^3}}$

Solution

a) We rewrite the expression as a quotient of two radicals. Then we simplify and multiply by 1:

$$\sqrt{\frac{4}{5b}} = \frac{\sqrt{4}}{\sqrt{5b}} = \frac{2}{\sqrt{5b}} \qquad \text{We assume } b > 0.$$

$$= \frac{2}{\sqrt{5b}} \cdot \frac{\sqrt{5b}}{\sqrt{5b}} \qquad \text{Multiplying by 1}$$

$$= \frac{2\sqrt{5b}}{(\sqrt{5b})^2} \qquad \text{Try to do this step mentally.}$$

$$= \frac{2\sqrt{5b}}{5b}.$$

b) To rationalize the denominator $\sqrt[3]{9x}$, note that $9x$ is $3 \cdot 3 \cdot x$. In order for this radicand to be a cube, we need another factor of 3 and two more factors of x. Thus we multiply by 1, using $\sqrt[3]{3x^2}/\sqrt[3]{3x^2}$:

$$\frac{\sqrt[3]{a}}{\sqrt[3]{9x}} = \frac{\sqrt[3]{a}}{\sqrt[3]{9x}} \cdot \frac{\sqrt[3]{3x^2}}{\sqrt[3]{3x^2}} \qquad \text{Multiplying by 1}$$

$$= \frac{\sqrt[3]{3ax^2}}{\sqrt[3]{27x^3}} \longleftarrow \text{This radicand is now a perfect cube.}$$

$$= \frac{\sqrt[3]{3ax^2}}{3x}.$$

c) To change the radicand $2x^2y^3$ into a perfect fifth power, we need four more factors of 2, three more factors of x, and two more factors of y. Thus we multiply by 1, using $\sqrt[5]{2^4x^3y^2}/\sqrt[5]{2^4x^3y^2}$, or $\sqrt[5]{16x^3y^2}/\sqrt[5]{16x^3y^2}$:

$$\frac{3x}{\sqrt[5]{2x^2y^3}} = \frac{3x}{\sqrt[5]{2x^2y^3}} \cdot \frac{\sqrt[5]{16x^3y^2}}{\sqrt[5]{16x^3y^2}} \quad \text{Multiplying by 1}$$

$$= \frac{3x\sqrt[5]{16x^3y^2}}{\sqrt[5]{32x^5y^5}} \quad \text{This radicand is now a perfect fifth power.}$$

$$= \frac{3x\sqrt[5]{16x^3y^2}}{2xy} = \frac{3\sqrt[5]{16x^3y^2}}{2y}. \quad \text{Always simplify if possible.}$$

Rationalize each numerator: (**a**) $\sqrt{\frac{7}{5}}$; (**b**) $\frac{\sqrt[3]{4a^2}}{\sqrt[3]{5b}}$.

Solution

a) $$\sqrt{\frac{7}{5}} = \sqrt{\frac{7}{5} \cdot \frac{7}{7}} \quad \text{Multiplying by 1 under the radical. We also could have multiplied by } \sqrt{7}/\sqrt{7} \text{ outside the radical.}$$

$$= \sqrt{\frac{49}{35}} \quad \text{The numerator is now a perfect square.}$$

$$= \frac{\sqrt{49}}{\sqrt{35}} \quad \text{Using the quotient rule for radicals}$$

$$= \frac{7}{\sqrt{35}}$$

b) $$\frac{\sqrt[3]{4a^2}}{\sqrt[3]{5b}} = \frac{\sqrt[3]{4a^2}}{\sqrt[3]{5b}} \cdot \frac{\sqrt[3]{2a}}{\sqrt[3]{2a}} \quad \text{Multiplying by 1}$$

$$= \frac{\sqrt[3]{8a^3}}{\sqrt[3]{10ba}} \quad \text{This radicand is now a perfect cube.}$$

$$= \frac{2a}{\sqrt[3]{10ab}}$$

Expressions Containing Several Radical Terms

Simplify by combining like radical terms, if possible.

a) $3\sqrt{8} - 5\sqrt{2}$

b) $9\sqrt{5} - 4\sqrt{3}$

c) $\sqrt[3]{2x^6y^4} + 7\sqrt[3]{2y}$

Solution

a)
$$
\begin{aligned}
3\sqrt{8} - 5\sqrt{2} &= 3\sqrt{4 \cdot 2} - 5\sqrt{2} \\
&= 3\sqrt{4} \cdot \sqrt{2} - 5\sqrt{2} \\
&= 3 \cdot 2 \cdot \sqrt{2} - 5\sqrt{2} \qquad \text{Simplifying } \sqrt{8} \\
&= 6\sqrt{2} - 5\sqrt{2} \\
&= \sqrt{2} \qquad \text{Combining like radicals}
\end{aligned}
$$

b) $9\sqrt{5} - 4\sqrt{3}$ cannot be simplified.

c)
$$
\begin{aligned}
\sqrt[3]{2x^6y^4} + 7\sqrt[3]{2y} &= \sqrt[3]{x^6y^3 \cdot 2y} + 7\sqrt[3]{2y} \\
&= \sqrt[3]{x^6y^3} \cdot \sqrt[3]{2y} + 7\sqrt[3]{2y} \qquad \text{Simplifying } \sqrt[3]{2x^6y^4} \\
&= x^2y \cdot \sqrt[3]{2y} + 7\sqrt[3]{2y} \\
&= (x^2y + 7)\sqrt[3]{2y} \qquad \text{Factoring to combine like radical terms}
\end{aligned}
$$

Products and Quotients of Two or More Radical Terms

Radical expressions often contain factors that have more than one term. The procedure for multiplying out such expressions is similar to finding products of polynomials. Some products will yield like radical terms, which we can now combine.

Rationalize each denominator: **(a)** $\dfrac{4}{\sqrt{3} + x}$; **(b)** $\dfrac{4 + \sqrt{2}}{\sqrt{5} - \sqrt{2}}$.

Solution

a) $\dfrac{4}{\sqrt{3}+x} = \dfrac{4}{\sqrt{3}+x} \cdot \dfrac{\sqrt{3}-x}{\sqrt{3}-x}$ Multiplying by 1, using the conjugate of $\sqrt{3}+x$, which is $\sqrt{3}-x$

$= \dfrac{4(\sqrt{3}-x)}{(\sqrt{3}+x)(\sqrt{3}-x)}$ Multiplying numerators and denominators

$= \dfrac{4(\sqrt{3}-x)}{(\sqrt{3})^2-x^2}$ Using FOIL in the denominator

$= \dfrac{4\sqrt{3}-4x}{3-x^2}$ Simplifying

b) $\dfrac{4+\sqrt{2}}{\sqrt{5}-\sqrt{2}} = \dfrac{4+\sqrt{2}}{\sqrt{5}-\sqrt{2}} \cdot \dfrac{\sqrt{5}+\sqrt{2}}{\sqrt{5}+\sqrt{2}}$ Multiplying by 1, using the conjugate of $\sqrt{5}-\sqrt{2}$, which is $\sqrt{5}+\sqrt{2}$

$= \dfrac{(4+\sqrt{2})(\sqrt{5}+\sqrt{2})}{(\sqrt{5}-\sqrt{2})(\sqrt{5}+\sqrt{2})}$ Multiplying numerators and denominators

$= \dfrac{4\sqrt{5}+4\sqrt{2}+\sqrt{2}\,\sqrt{5}+(\sqrt{2})^2}{(\sqrt{5})^2-(\sqrt{2})^2}$ Using FOIL

$= \dfrac{4\sqrt{5}+4\sqrt{2}+\sqrt{10}+2}{5-2}$ Squaring in the denominator and the numerator

$= \dfrac{4\sqrt{5}+4\sqrt{2}+\sqrt{10}+2}{3}$

To Simplify Products or Quotients with Differing Indices

1. Convert all radical expressions to exponential notation.
2. When the bases are identical, subtract exponents to divide and add exponents to multiply. This may require finding a common denominator.
3. Convert back to radical notation and, if possible, simplify.

If $f(x) = \sqrt[3]{x^2}$ and $g(x) = \sqrt{x} + \sqrt[4]{x}$, find $(f \cdot g)(x)$.

Solution Recall from Section 7.4 that $(f \cdot g)(x) = f(x) \cdot g(x)$. Thus,

$$
\begin{aligned}
(f \cdot g)(x) &= \sqrt[3]{x^2}\left(\sqrt{x} + \sqrt[4]{x}\right) && x \text{ is assumed to be nonnegative.} \\
&= x^{2/3}(x^{1/2} + x^{1/4}) && \text{Converting to exponential notation} \\
&= x^{2/3} \cdot x^{1/2} + x^{2/3} \cdot x^{1/4} && \text{Using the distributive law} \\
&= x^{2/3+1/2} + x^{2/3+1/4} && \text{Adding exponents} \\
&= x^{7/6} + x^{11/12} && \tfrac{2}{3} + \tfrac{1}{2} = \tfrac{4}{6} + \tfrac{3}{6};\ \tfrac{2}{3} + \tfrac{1}{4} = \tfrac{8}{12} + \tfrac{3}{12} \\
&= \sqrt[6]{x^7} + \sqrt[12]{x^{11}} && \text{Converting back to radical notation} \\
&= \sqrt[6]{x^6}\,\sqrt[6]{x} + \sqrt[12]{x^{11}} && \text{Simplifying} \\
&= x\sqrt[6]{x} + \sqrt[12]{x^{11}}
\end{aligned}
$$

Radical Equations

To Solve an Equation with a Radical Term

1. Isolate the radical term on one side of the equation.
2. Use the principle of powers and solve the resulting equation.
3. Check any possible solution in the original equation.

Solve: $x = \sqrt{x + 7} + 5$.

Solution

$$x = \sqrt{x+7} + 5$$

$$x - 5 = \sqrt{x+7}$$ Subtracting 5 from both sides. This isolates the radical term.

$$(x-5)^2 = \left(\sqrt{x+7}\right)^2$$
$$x^2 - 10x + 25 = x + 7$$ Using the principle of powers; squaring both sides

$$x^2 - 11x + 18 = 0$$ Adding $-x - 7$ to both sides to write the quadratic equation in standard form

$$(x-9)(x-2) = 0$$ Factoring

$$x = 9 \quad or \quad x = 2$$ Using the principle of zero products

The possible solutions are 9 and 2. Let's check.

Check:

For 9:

$$x = \sqrt{x+7} + 5$$
$$9 \;?\; \sqrt{9+7} + 5$$
$$9 \;|\; 9$$ TRUE

For 2:

$$x = \sqrt{x+7} + 5$$
$$2 \;?\; \sqrt{2+7} + 5$$
$$2 \;|\; 8$$ FALSE

Since 9 checks but 2 does not, the solution is 9.

It is important to isolate a radical term before using the principle of powers. Suppose in Example 3 that both sides of the equation were squared *before* isolating the radical. We then would have had the expression $\left(\sqrt{x+7} + 5\right)^2$ or $x + 7 + 10\sqrt{x+7} + 25$ on the right side, and the radical would have remained in the problem.

Solve: $\sqrt{2x-5} = 1 + \sqrt{x-3}$.

Solution

$$\sqrt{2x-5} = 1 + \sqrt{x-3}$$

$$\left(\sqrt{2x-5}\right)^2 = \left(1 + \sqrt{x-3}\right)^2$$

One radical is already isolated. We square both sides.

This is like squaring a binomial. We square 1, then find twice the product of 1 and $\sqrt{x-3}$ and then the square of $\sqrt{x-3}$.

$$2x-5 = 1 + 2\sqrt{x-3} + \left(\sqrt{x-3}\right)^2$$

$$2x-5 = 1 + 2\sqrt{x-3} + (x-3)$$

$$x-3 = 2\sqrt{x-3}$$

Isolating the remaining radical term

$$(x-3)^2 = \left(2\sqrt{x-3}\right)^2$$

Squaring both sides

$$x^2 - 6x + 9 = 4(x-3)$$

Remember to square both the 2 and the $\sqrt{x-3}$ on the right side.

$$x^2 - 6x + 9 = 4x - 12$$

$$x^2 - 10x + 21 = 0$$

$$(x-7)(x-3) = 0$$

Factoring

$$x = 7 \quad or \quad x = 3$$

Using the principle of zero products

References:

1. Elementary and Intermediate Algebra, Concepts and Applications: A combined Approach, First Edition; Marvin L. Bittinger, David L. Ellenbogen, and Barbara Johnson; Addison Wesley Longman Inc., 1996.

2. Beginning and Intermediate Algebra, Third Edition; Margaret L. Lial, John Hornsby, and Terry McGinnis; Pearson Education, Inc., 2004.