Quadratic and Rational (Nonlinear) Inequalities

Let's review linear inequalities and remember some important details so that we can apply them to the nonlinear inequalities.

A linear inequality is one in which the variable has a power of one, such as: $5-3x \le 23$
To solve this inequality:

Isolate the x term: $-3x \le 18$

Divide by -3: $\frac{-3x}{-3} \le \frac{18}{-3} \quad \rightarrow \quad x \ge -6$

Remember, division (or multiplication) by a negative number switches the inequality sign.

The solutions to the inequality are numbers -6 or greater. As long as you choose numbers in this solution set, they will produce a true statement in the inequality.

Pick some numbers that are greater than or equal to -6, and verify they give a true statement.

Choose x = -5.

$5-3(-5) \overset{?}{\le} 23$

$5+15 \overset{?}{\le} 23$

$20 \overset{?}{\le} 23$

True! So x = -5 is one of the many solutions to the inequality.

Choose x = 4.

$5-3(4) \overset{?}{\le} 23$

$5-12 \overset{?}{\le} 23$

$-7 \overset{?}{\le} 23$

True! So x = 4 is one of the many solutions to the inequality.

Now select something that was NOT in the solution set of $x \ge -6$, say, x = -10.

$5-3(-10) \overset{?}{\le} 23$

$5+30 \overset{?}{\le} 23$

$35 \overset{?}{\le} 23$

False!!! Therefore the number -10 is not in the solution set.

What about problems such as:

A) $x^2 - 16 \leq 0$ B) $\frac{x-3}{x+5} > 0$

These are not linear inequalities and they are not solved the same way. You must be able to identify what makes an inequality nonlinear.

Identification of a Nonlinear Inequality

Problem A has a power of two, which makes it a quadratic, and B has an x in the denominator. These two facts make A and B nonlinear inequalities, and thus will be solved differently than linear inequalities.

If you determine the inequality is LINEAR (exponent of one, not a rational function) then solve by isolating x.

If you determine the inequality is NONLINEAR, which means it will have a power larger than one, OR an x in the denominator, then solve as follows.

METHOD OF SOLVING NONLINEAR INEQUALITIES

Step 1: Manipulate the inequality so that the right side of the inequality is zero.

Step 2: From Step 1, take note whether the inequality is $<$, $\leq$, $>$ or $\geq$.

Step 3: Factor both the numerator and denominator.

Step 4: Determine any numbers that could produce division by zero and make a note that this number can NOT be part of the solution set. (This step only applies when there is a denominator in the problem.)

Step 5: Find all the *critical numbers*. (These are the numbers which could produce zero in either the numerator or denominator.) Set all factors = 0 and solve each.

Step 6: Plot these critical numbers in numeric order on the number line.

Step 7: For each interval created by the critical numbers, evaluate any number in the interval to determine whether the expression is positive or negative in that interval.

Step 8: Look back at Step 2. If Step 2 was $\geq$ or $>$, write the intervals that are positive. If Step 2 was $\leq$ or $<$, write the intervals that are negative.

Step 9: If the inequality is $\leq$ or $\geq$ (includes an equality), check the critical numbers of the inequality (the end points of the intervals). Remember, the zeros of the denominator can never be part of the solution. The zeros of the numerator are included in the solution set.

Example: Solve $x^2 - 4 \leq 0$

At first glance, perhaps you would like to move the 4 to the other side: $x^2 \leq 4$.

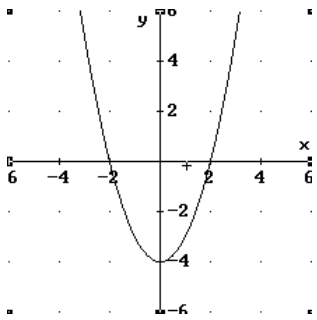
Now what? Can you take the square root of both sides?

The answer is NO! This is a very common mistake to make. You may only take the square root of both sides when you have *only* an equal sign, not an inequality.

So what do we do? Let's think about this. We want to know for what values of x is $x^2 - 4 \leq 0$, which is asking the same question as for what values of x is the parabola $f(x) = x^2 - 4$ less than or equal to zero?

In other words, when is the parabola on, or below the x-axis?

Let's graph it and take a look.



For what values of x is the parabola below or on the x-axis?

$-2 \leq x \leq 2$ is the set of all x's such that the graph is below or on the x-axis.

Graphing a nonlinear inequality can be cumbersome, so let's solve this by applying the steps listed above.

Step 1: This step is already done. The right side does equal zero.

Step 2: The inequality is $\leq$ so you will look for the negatives in Step 8.

Step 3: Factor: $(x + 2)(x - 2) \leq 0$

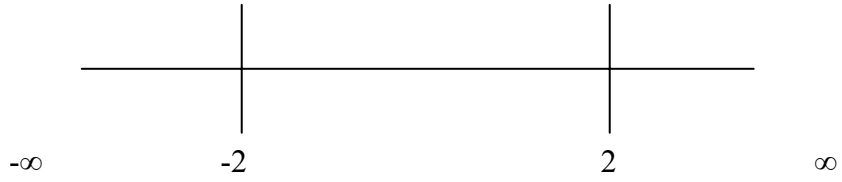
Step 4: There is no denominator, so there is nothing to note here.

Step 5: Find all the critical values. (This is where the left hand side could produce zero in either the numerator or denominator.)

Set all factors = 0 $\rightarrow$ $x + 2 = 0$ or $x - 2 = 0$

$x = 2$ and $x = -2$ are the critical values.

Step 6:



Step 7: Each interval must be checked and determined if it is positive or negative.

Choose any number in each interval and use these values in the factored form from Step 2, $(x + 2)(x – 2)$, to determine whether the inequality is positive or negative for each interval.

Interval on the Far Left
Any number to the left of -2 can be used, say -3, so evaluating:
$(-3 + 2)(-3 – 2) = (-1)(-5) = 5$ which is positive.

Middle Interval
Any number between -2 and 2 can be used, say 0, so evaluating:
$(0 + 2)(0 – 2) = (2)(-2) = -2$, which is negative.

Interval on the Far Right
Any number to the right of 2 can be used, say +3, so evaluating:
$(3 + 2)(3 – 2) = (5)(1) = 5$ which is positive.

Step 8: Look back at Step 2; our question is $\leq$ so we write the intervals that are negative. (These are the intervals that the parabola dips below the x-axis.)

The solution so far is: $-2 < x < 2$

Step 9: The inequality symbol is $\leq$ so we want the two critical values which also make the equality true: 2 and -2.

The solution is: $-2 \leq x \leq 2$

Example: Solve $x^2 - 2x > 3$

This is nonlinear because the variable has a power larger than one.

Step 1: Set the right side of the inequality equal to zero: $x^2 - 2x - 3 > 0$

Step 2: The inequality is $>$ so look for the positives in Step 8.

Step 3: Factor: $(x + 1)(x - 3) > 0$

Step 4: There is no denominator, so skip this step.

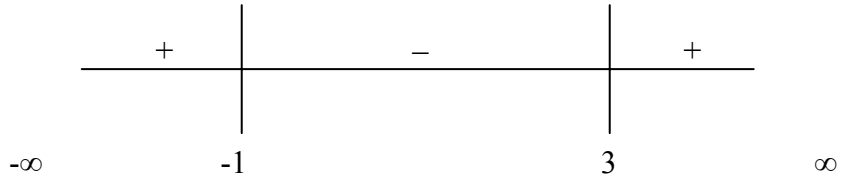
Step 5: Critical values are:

$x + 1 = 0$ or $x - 3 = 0$
$x = -1$ or $x = 3$

Step 6:

Step 7: The following numbers may be used to check each interval. They could have also been numbers such as -2, l, 4. As long as they fall within each interval, they are valid numbers to use. Note that we only care whether the result is positive or negative.

Number to Check	Plug into Equation	Sign of each factor	Product
-3	(-3 + 1)(-3 – 3)	(–)(–)	(+)
0	(0 + 1)(0 – 3)	(+)(–)	(–)
6	(6 + 1)(6 – 3)	(+)(+)	(+)



Step 8: Look back at Step 2; our question is $>$, so we write the intervals that are positive. The solution is $x < -1$ or $x > 3$.

Step 9: Doesn't apply since the problem did not include equality.

Let's determine what this means.

If we choose numbers that are in the solution set and plug them back into the inequality, we will come up with a true statement. In this case we will get numbers that are greater than zero.

Choose some numbers that are IN the solution set. Say -7, +6

Verify -7 and 6 are two of the many solutions for this inequality.

$(-7)^2 - 2(-7) - 3 \stackrel{?}{>} 0$

$49 + 14 - 3 \stackrel{?}{>} 0$

$60 \stackrel{?}{>} 0$

TRUE!!

$(6)^2 - 2(6) - 3 \stackrel{?}{>} 0$

$36 - 12 - 3 \stackrel{?}{>} 0$

$21 \stackrel{?}{>} 0$

TRUE!!

Now choose values that are NOT in the solution set and verify that a false statement is produced. The values of 0 and 1 are NOT in the solution set, and thus will produce a false statement.

$(0)^2 - 2(0) - 3 \stackrel{?}{>} 0$

$-3 \stackrel{?}{>} 0$

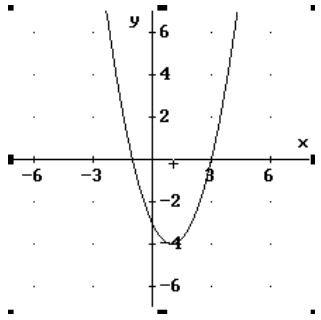
FALSE!!

This verifies that 0 is not a solution for the inequality. (You can also verify that 1 is NOT a solution.)

Let's look at the graph of this problem and see the solution set.

The problem was $x^2 - 2x - 3 > 0$ and the solution set is $x < -1$ or $x > 3$.

The inequality > is asking "for what values of x is the parabola ABOVE the x-axis?"



On the graph above you can see that the graph is ABOVE (>) the x-axis when x is greater than 3, or when x is less than -1.

Look at the graph and the numbers that we checked, x = -7 and x = 6.

When x = 6 the graph is above the x-axis. This is also true when x = -7. Now look at the number zero. We verified that zero was NOT in the solution set, and if you look at the graph at x = 0, the graph is NOT above the x-axis.

***If the question had been less than, as in $x^2 - 2x - 3 < 0$, then the solution set would have been $-1 < x < 3$; this is exactly where the graph is below (<) the x-axis.

Example: Solve $x^3 < x$

This is nonlinear since it has a power greater than one.

At first glance you may want to divide both sides by x. This is a very common and serious mistake. You can *never* solve any type of equation for x, inequality or equality, by dividing by the variable.

Step 1: Set the right side of the inequality equal to zero: $x^3 - x < 0$

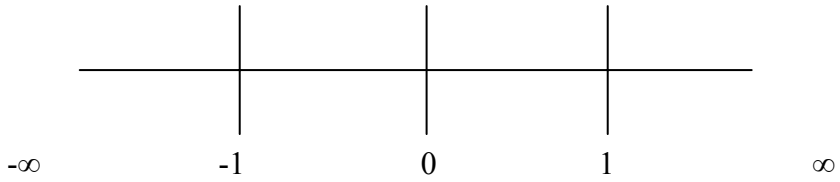
Step 2: The inequality is <, so you will be looking for the negatives.

Step 3: Factor: $x(x^2 - 1) < 0 \rightarrow x(x + 1)(x - 1) < 0$

Step 4: There is no denominator, so skip this step.

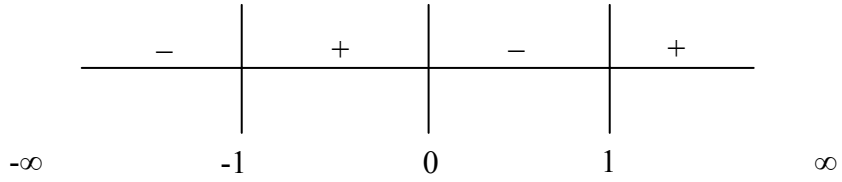
Step 5: The critical values are x = 0, x = 1, x = -1.

Step 6:



Step 7:

Number to Check	Plug into Equation	Sign of each factor	Product
-2	$(-2)[(-2)^2 - 1]$	(–)(+)	(–)
-0.5	$(-0.5)[(-0.5)^2 - 1]$	(–)(–)	(+)
0.5	$(0.5)[(0.5)^2 - 1]$	(+)(–)	(–)
2	$(2)[(2)^2 - 1]$	(+)(+)	(+)



Step 8: From Step 2, we know the inequality sign is <, so we need to write the intervals that are negative (–). This is where the graph is below the x-axis.

The solution is $x < -1$ or $0 < x < 1$.

Step 9: Doesn't apply since the problem did not include equality.

***If the question would instead have been greater or equal to, $x^3 \geq x$, the solution would have the positive intervals. The solution would have been $-1 \leq x \leq 0$ or $x \geq 1$.

Example: Solve $\frac{x-3}{x+2} \geq 0$

Since there is an x in the denominator this problem is nonlinear, and therefore must be solved by methods of solving nonlinear inequalities.

Step 1: This step is already complete.

Step 2: The inequality is $\geq$ so look for the positives in Step 8.

Step 3: This step is already complete.

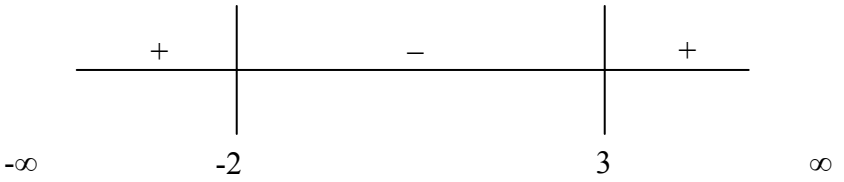
Step 4: When x = -2, division by zero will occur, and therefore x = -2 can NOT be included in the solution set.

Step 5: Critical values are x = -2 and x = 3.

Step 6:

Step 7:

Number to Check	Plug into Equation	Sign of each factor	Product
-3	$\frac{-3-3}{-3+2}$	$\frac{(-)}{(-)}$	(+)
0	$\frac{0-3}{0+2}$	$\frac{(-)}{(+)}$	(–)
4	$\frac{4-3}{4+2}$	$\frac{(+)}{(+)}$	(+)



Step 8: From Step 2, the inequality is $\geq$ and so we must write the positive intervals:
$x < -2$ or $x > 3$.

Step 9: From Step 4 we found the number -2 could not be considered part of the solution set because it is not in the domain. But since we have equality, the zero of the numerator is part of the solution.

The solution is $x < -2$ or $x \geq 3$.

Example: Solve $\dfrac{3x}{x-4} \leq 2$

Again there is an x in the denominator, so this problem is nonlinear.

Step 1: Subtract 2 from both sides.

$\dfrac{3x}{x-4} - 2 \leq 0$ Simplify, using the lowest common denominator.

$\dfrac{3x-2(x-4)}{x-4} \leq 0 \quad \rightarrow \quad \dfrac{x+8}{x-4} \leq 0$ This is the form we concentrate on.

Step 2: The inequality is $\leq$ so the negative intervals will be found in Step 8.

Step 3: This step is already complete.

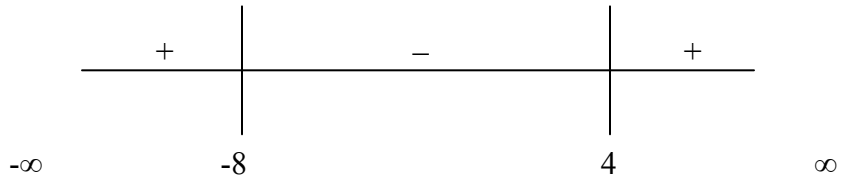
Step 4: The number $x = 4$ will produce division by zero, so it may NOT be included in the solution set.

Step 5: The critical values are $x = -8$ and $x = 4$.

Step 6:

Step 7:

Number to Check	Plug into Equation	Sign of each factor	Product
-9	$\dfrac{-9+8}{-9-4}$	$\dfrac{(-)}{(-)}$	(+)
0	$\dfrac{0+8}{0-4}$	$\dfrac{(+)}{(-)}$	(–)
5	$\dfrac{5+8}{5-4}$	$\dfrac{(+)}{(+)}$	(+)



Step 8: From Step 2, the inequality is $\leq$, so the interval that is negative is our solution set is $-8 < x < 4$.

Step 9: From Step 4, the number 4 cannot be part of the solution, but -8 is since it makes the expression zero. So the solution is $-8 \leq x < 4$.

***If the question had been greater than or equal to, $\frac{x+8}{x-4} \geq 0$, the solution would have been $x \leq -8$ or $x > 4$.

Problems - Solve:

1) $4x^2 - 25 \leq 0$

2) $3x^2 + 13x \geq 10$

3) $x^2 - 5x < 5$

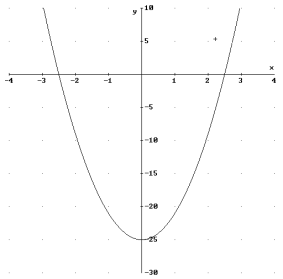
4) $2x^3 + 7x^2 + 6x > 0$

5) $\frac{x-3}{x+1} > 4$

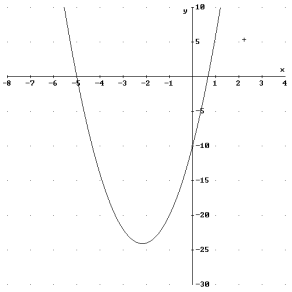
6) $\frac{5x-8}{x^2-9} \geq 0$

Answers

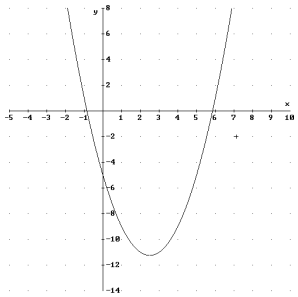
1) $-\frac{5}{2} \leq x \leq \frac{5}{2}$



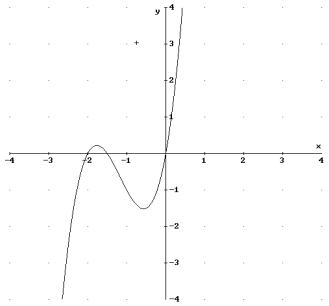
2) $x \leq -5$ or $x \geq \frac{2}{3}$



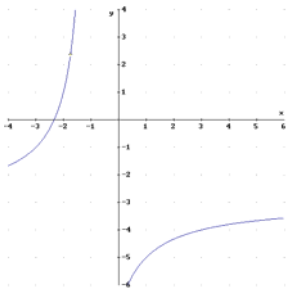
3) $\frac{5-3\sqrt{5}}{2} < x < \frac{5+3\sqrt{5}}{2}$



4) $-2 < x < -\frac{3}{2}$ or $x > 0$



5) $-\frac{7}{3} < x < -1$



6) $-3 < x \leq \frac{8}{5}$ or $x > 3$