Multiplying and Dividing Rational Expressions (6.2)

Multiplication

Multiplying rational expressions is pretty straightforward – you multiply the numerators together to make a new numerator and you multiply the denominators together to make a new denominator. In generic format it would look like this: $\frac{A}{B} \bullet \frac{C}{D} = \frac{AC}{BD}$.

Another way to remember this is to "multiply straight across": $\frac{A}{B} \overset{\bullet}{\underset{\bullet}{\frac{\rightarrow}{\rightarrow}}} \frac{C}{D} = \frac{AC}{BD}$ or $\frac{(A)(C)}{(B)(D)}$

For example: $$\frac{x}{5} \bullet \frac{x^2+2}{x+1} = \frac{x(x^2+2)}{5(x+1)}$$

After multiplication, check to see if the result can be *simplified*:

> Are both the numerator and the denominator expressed strictly as products of factors? Does the exact same factor appear in both the numerator and denominator (and therefore can be canceled)?

Example 1. Multiply and simplify $\frac{x+3}{x+2} \bullet \frac{x-1}{x+3}$

Step 1. Multiply numerators and multiply denominators: $\frac{x+3}{x+2} \bullet \frac{x-1}{x+3} = \frac{(x+3)(x-1)}{(x+2)(x+3)}$

The new rational expression consists of products of factors in both the numerator and denominator.

Step 2. Check for factors common to both the numerator and the denominator and cancel them to simplify.

$$\frac{\cancel{(x+3)}(x-1)}{(x+2)\cancel{(x+3)}} = \frac{x-1}{x+2}$$

Sometimes, the numerators or denominators have polynomials in them that are factorable. To fully explore all opportunities for canceling and simplifying, you must first factor the polynomials and then compare the factors on the top and the bottom.

Example 3. Multiply and simplify $\frac{3x^2-27}{x^2-x-12} \bullet \frac{x^2-8x+16}{5x-20}$

Step 1. Multiply numerators and multiply denominators: $\frac{3x^2-27}{x^2-x-12} \bullet \frac{x^2-8x+16}{5x-20}$

$$= \frac{(3x^2-27)(x^2-8x+16)}{(x^2-x-12)(5x-20)}$$

Step 2. Factor completely.

Numerator $\begin{cases} 3x^2-27 = 3(x^2-9) = \mathbf{3(x+3)(x-3)} \\ x^2-8x+16 = \mathbf{(x-4)(x-4)} \end{cases}$

Denominator $\begin{cases} x^2-x-12 = \mathbf{(x+3)(x-4)} \\ 5x-20 = \mathbf{5(x-4)} \end{cases}$

$$\rightarrow \frac{(3)(x+3)(x-3)(x-4)(x-4)}{(x+3)(x-4)(5)(x-4)}$$

Step 3. Cancel common factors.

$$= \frac{(3)\cancel{(x+3)}(x-3)\cancel{(x-4)}\cancel{(x-4)}}{\cancel{(x+3)}\cancel{(x-4)}(5)\cancel{(x-4)}}$$

$$= \frac{3(x-3)}{5}$$

Division

Dividing rational expressions is also straightforward - there's just one more step involved.

Recall that to divide by a fraction, you *multiply* by its *reciprocal*: $\frac{1}{2} \div \frac{3}{5} = \frac{1}{2} \bullet \frac{5}{3} = \frac{(1)(5)}{(2)(3)} = \frac{5}{6}$

or, generically: $\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \bullet \frac{D}{C} = \frac{AD}{BC}$. So, once you "flip over" the expression you are *dividing by*, proceed as above by multiplying, factoring, and canceling.

Example 4. Divide and simplify $\dfrac{x}{x-y} \div \dfrac{y-1}{x-y}$

Step 1. Invert the second expression (the divisor) and multiply.

$$\frac{x}{x-y} \div \frac{y-1}{x-y} = \frac{x}{x-y} \bullet \frac{x-y}{y-1} = \frac{x(x-y)}{(x-y)(y-1)}$$

Step 2. Check to see if you can factor anything further. You can't, so now check for common factors and cancel them to simplify:

$$\frac{x\cancel{(x-y)}}{\cancel{(x-y)}(y-1)} = \frac{x}{y-1}$$

Example 5. Divide and simplify $\dfrac{x^2+x-30}{x^2+7x+12} \div \dfrac{x^2-6x+5}{x^2+6x+9}$

Step 1. Invert the divisor and multiply.

$$\frac{x^2+x-30}{x^2+7x+12} \div \frac{x^2-6x+5}{x^2+6x+9} = \frac{x^2+x-30}{x^2+7x+12} \bullet \frac{x^2+6x+9}{x^2-6x+5}$$

$$= \frac{(x^2+x-30)(x^2+6x+9)}{(x^2+7x+12)(x^2-6x+5)}$$

Step 2. Factor completely.

Numerator $\begin{cases} x^2 + x - 30 = \mathbf{(x-5)(x+6)} \\ x^2 + 6x + 9 = \mathbf{(x+3)(x+3)} \end{cases}$

Denominator $\begin{cases} x^2 + 7x + 12 = \mathbf{(x+3)(x+4)} \\ x^2 - 6x + 5 = \mathbf{(x-1)(x-5)} \end{cases}$

$$\rightarrow \frac{(x-5)(x+6)(x+3)(x+3)}{(x+3)(x+4)(x-1)(x-5)}$$

Step 3. Cancel common factors.

$$= \frac{\cancel{(x-5)}(x+6)\cancel{(x+3)}(x+3)}{\cancel{(x+3)}(x+4)(x-1)\cancel{(x-5)}}$$

$$= \frac{(x+6)(x+3)}{(x+4)(x-1)}$$

Problems

Multiply and simplify:

1. $\frac{24}{x^5} \bullet \frac{x^2}{3}$

2. $\frac{14x^2y}{4} \bullet \frac{2}{xy^3}$

3. $\frac{x^2}{10x-2} \bullet \frac{2}{x^3+x^2}$

4. $\frac{x^2-9}{x+1} \bullet \frac{x^2+2x+1}{2x^2-10x+12}$

5. $\frac{x^2+4}{x^2-6x+9} \bullet \frac{x^2+5x+6}{x^4-16}$

Divide and simplify:

6. $\frac{x}{y^2} \div \frac{x^2}{y^3}$

7. $\frac{x+6}{4} \div \frac{x}{2}$

8. $\frac{4-2x}{12} \div \frac{x-2}{2}$

9. $\frac{x^2-8x+16}{x^2+8x+16} \div \frac{(x-4)^4}{(x+4)^4}$

10. $\frac{a^4-81b^4}{a^2c-6abc+9b^2c} \div \frac{a^2+6ab+9b^2}{(a-3b)^2}$

Answers

1. $\frac{8}{x^3}$

2. $\frac{7x}{y^2}$

3. $\frac{1}{(5x-1)(x+1)}$

4. $\frac{(x+3)(x+1)}{2(x-2)}$

5. $\frac{(x+3)}{(x-3)^2(x-2)}$

6. $\frac{y}{x}$

7. $\frac{x+6}{2x}$

8. $\frac{-1}{3}$

9. $\frac{(x+4)^2}{(x-4)^2}$

10. $\frac{(a^2+9b^2)(a-3b)}{c(a+3b)}$