16 Factoring (5.1 – 5.3)

Factoring is the reverse of multiplying. When we multiplied monomials or polynomials together, we got a new monomial or a string of monomials that were added (or subtracted) together. For example, if we multiply $5x^2$ by $2y$, we get $10x^2y$. So $5x^2$ and $2y$ can be said to be factors of $10x^2y$ – they are terms that when multiplied together produce $10x^2y$. Factoring "rewrites" an expression as terms that are multiplied. Let's look at different ways of factoring.

Factoring Monomials

Monomials can often be factored in more than one way.

Example 1 : Factor $27x^3$

Possible factors are: $3x^2 \cdot 9x$

or $9x^2 \cdot 3x$

or $27x \cdot x^2$

or $27 \cdot x^3$ All of these will produce $27x^3$ when multiplied

etc.

Factoring Out a Common Factor

When we multiplied a polynomial by a monomial, we used the distributive law:

$$A(B + C) = AB + AC.$$

For example:

$$\mathbf{2x}(x^2 + 3x - 5) = \mathbf{2x} \cdot x^2 + \mathbf{2x} \cdot 3x - \mathbf{2x} \cdot 5 = 2x^3 + 6x^2 - 10x$$

Since every term of $(x^2 + 3x - 5)$ was multiplied by the same thing $(2x)$, $2x$ is called a common factor of the expression $2x^3 + 6x^2 - 10x$ and can be taken back out to express the polynomial as a product.

The first question to ask when faced with factoring a polynomial is – Are there any common factors? Then "take out" the *largest common factor* you can find.

Example 2: Factor $18x^5 + 45x^3 + 81x^2$

Step 1:

Do all the terms in the polynomial have the same variable (like x or y)? $18\mathbf{x}^5 + 45\mathbf{x}^3 + 81\mathbf{x}^2$

If yes, factor each term in the expression so that one of the factors in each term is the variable with the lowest power attached to it. $18x^3 \cdot \mathbf{x^2} + 45x \cdot \mathbf{x^2} + 81 \cdot \mathbf{x^2}$

Then take out this common factor. $\mathbf{x^2}(18x^3 + 45x + 81)$

what's left is still added together

Step 2:

Do the numbers in the expression have a common factor?

This may take some trial and error, so if it's not obvious, start dividing the numbers by something small, like 2 or 3 or 5, until they are completely factored.

$$x^2(\overset{(18)}{2\cdot\mathbf{9}}\, x^3 + \overset{(45)}{5\cdot\mathbf{9}}\, x + \overset{(81)}{9\cdot\mathbf{9}})$$

If the same factor appears in all of the terms, it is a common factor and can be "taken out". This expression cannot be factored any further.

$\mathbf{9}x^2(2x^3 + 5x + 9)$

Factoring by Grouping

Sometimes the common factor is a polynomial. Don't let this throw you! No matter how complex an expression may be, if the <u>same expression is a factor (a multiplier) in all of the terms</u> in the polynomial, then factor it out.

For example, we know we can factor out the common term x from the expression $x^2 + x$ and create the product $x(x + 1)$.

What about: $x\mathbf{(x + 1)} + 5\mathbf{(x + 1)}$?

The term (x+1) is <u>a factor of each term</u> of the expression and is therefore a *common factor*.

When we factor it out, we get: $\mathbf{(x+1)}(x+5)$.

↖ what's left is still added together

If you have a polynomial with no factors common to <u>all</u> the terms, try "grouping" the terms in a way that will let you factor each group. Then you might end up actually creating a factor common to both groups. Here's how it works.

Example 3: Factor by grouping $6x^3 – 3x^2 + 2xy – y$

Step 1. There are no factors common to all the terms. Group the first two terms together and the last two terms together.

$6x^3 – 3x^2 + 2xy – y$

$(6x^3 – 3x^2) + (2xy – y)$

Step 2. Factor each group separately. Take out the common variable with the lowest power and any common numerical factors.

$(2x\cdot\mathbf{3x^2} – \mathbf{3x^2}) + (2x\cdot\mathbf{y} – \mathbf{y})$

$\mathbf{3x^2}(2x-1) + \mathbf{y}(2x-1)$

Step 3. Now look again for common factors. You've created a new common factor: (2x-1) !

$3x^2\mathbf{(2x-1)} + y\mathbf{(2x-1)}$

Step 4. Factor out the common polynomial and add the remaining terms together.

$(2x - 1)(3x^2 + y)$

CAUTION! If the polynomial had been written as $6x^3 – 3x^2 - 2xy + y$, when you do the grouping, you have to <u>change the sign</u> of the last term:

$6x^3 – 3x^2 - 2xy + y = (6x^3 – 3x^2) – (2xy – y)$

↑ └ **this sign changes**

because this negative sign
is now a <u>coefficient</u> of –1: - (2xy – y) = -2xy – (-y) = -2xy + y

Factoring Trinomials: $x^2 + bx + c$ (coefficient of 1 on the x^2)

When you multiply two binomials (using the FOIL method), the terms add up like this:

$$(x+3)(x+5) = x^2 + 5x + 3x + 15 = x^2 + 8x + 15$$

Notice that the <u>last term</u> in the trinomial is the <u>product</u> of the last terms in the binomials ($15 = 3 \cdot 5$). Now notice that the numerical part of the <u>middle term</u> in the trinomial is the <u>sum</u> of the last terms in the binomials ($8 = 3 + 5$). IF THE TRINOMIAL IS FACTORABLE, THIS IS ALWAYS TRUE. (<u>Important</u>! this is true only if the x^2 term is multiplied only by 1.)

Using this information, if you have a trinomial with no common factors, you can do a trial-and-error way of factoring. Always check and take out any common factors first!

Example 4: Factor $x^2 + 10x + 16$

Step 1. The last term is +16. The numerical part of the middle term is +10.

List possible factors of the last term:	$+16 =$	$1 \cdot 16$	$-1 \cdot -16$
		$2 \cdot 8$	$-2 \cdot -8$
		$4 \cdot 4$	$-4 \cdot -4$

Step 2. Test pairs of factors by <u>adding</u> them together.	$1+16 = 17$	$-1-16 = -17$
Does any set of factors add up to the middle term of +10? ↗	**$2+8 = 10$**	$-2-8 = -10$
We have a winner!	$4+4 = 8$	$-4-4 = -8$

Step 3. We can now write **$x^2 + 10x + 16$** as **$(x+2)(x+8)$.**

Step 4. Check your factoring by doing the multiplication: $(x+2)(x+8) = x^2 + 8x + 2x + 16$

$$= x^2 + 10x + 16$$ ☺

Notice that I could have gotten +16 by multiplying <u>two positive</u> numbers together or by multiplying <u>two negative</u> numbers together, but only the positive factors added up to +10. Here's some rules-of-thumb to figure out the signs in the binomial factors.

Last term <u>positive</u>
Middle term <u>positive</u>
→ the signs in both factors are positive

$$x^2 \underset{\substack{\uparrow \\ \text{pos}}}{+} 10x \underset{\substack{\uparrow \\ \text{pos}}}{+} 16 \quad \rightarrow \quad (x \underset{\substack{\uparrow \\ \text{pos}}}{+} 2)(x \underset{\substack{\uparrow \\ \text{pos}}}{+} 8)$$

Last term <u>positive</u>
Middle term <u>negative</u>
→ the signs in both factors are negative:
$(-2)(-8) = +16$
$-2 - 8 = -10$

$$x^2 \underset{\substack{\uparrow \\ \text{neg}}}{-} 10x \underset{\substack{\uparrow \\ \text{pos}}}{+} 16 \quad \rightarrow \quad (x \underset{\substack{\uparrow \\ \text{neg}}}{-} 2)(x \underset{\substack{\uparrow \\ \text{neg}}}{-} 8)$$

Last term <u>negative</u>
Middle term <u>positive</u>
→ the sign in the factor with the <u>larger number</u> will be <u>positive</u> and the sign in the other factor will be <u>negative</u>
.

$$x^2 \underset{\substack{\uparrow\\\text{pos}}}{+} 6x \underset{\substack{\uparrow\\\text{neg}}}{-} 16 \quad\rightarrow\quad (x \underset{\substack{\uparrow\\\text{neg}}}{-} 2)(x \underset{\substack{\uparrow\\\text{pos}\\\text{(larger number)}}}{+} 8)$$

Last term <u>negative</u>
Middle term <u>negative</u>
→ the sign in the factor with the <u>larger number</u> will be <u>negative</u> and the sign in the other factor will be <u>positive</u>.

$$x^2 \underset{\substack{\uparrow\\\text{neg}}}{-} 6x \underset{\substack{\uparrow\\\text{neg}}}{-} 16 \quad\rightarrow\quad (x \underset{\substack{\uparrow\\\text{pos}}}{+} 2)(x \underset{\substack{\uparrow\\\text{neg}\\\text{(larger number)}}}{-} 8)$$

Factoring Trinomials: $ax^2 + bx + c$ (coefficient of a number other than 1 on the x^2)

What if the x^2 term in the trinomial has a coefficient other than 1 in front of it?

Remember when you FOIL two binomials: $(2x + 3)(3x + 4)$
First term · First term = First term of the trinomial $2x \cdot 3x = 6x^2$
Outer term · Outer term + Inner term · Inner term = Middle term of the trinomial $\mathbf{2x \cdot 4 + 3x \cdot 3 = 8x + 9x = 17x}$
Last term · Last term = Last term of the trinomial $3 \cdot 4 = 12$
$(2x + 3)(3x + 4) = 6x^2 + 17x + 12$

Let's look at: $9x^2 + 18x + 8$.

Now we can't just look at the factors of the last term (8) to see if they add up to 18, because that 9 in front of the x^2 is also going to be involved in calculating the middle term.

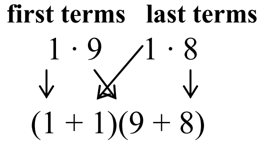
One method to resolve our dilemma is **trial-and-error**.

Step 1. Factor <u>both</u> the 9 in the first term and the 8 of the last term. Line up your factors next to each other.

<u>9</u>	<u>8</u>
1 · 9	1 · 8
3 · 3	2 · 4

The first column contains possible numbers of the first terms of the binomials that our trinomial will (hopefully) factor into. The second column contains possible numbers of the last terms of the binomials.

Step 2. Picture the trinomial in factored form: (? + ?)(? + ?)
Distribute the factors of the 9 to the first term in each binomial factor; distribute the factors of the 8 to the last term in each binomial factor.



Step 3. To get the middle term of the trinomial, multiply only the outer terms and the inner terms and add the products. Do any add up to our middle term of 18?

Outer Inner

$(1 + 1)(9 + 8) \rightarrow 1 \cdot 8 + 1 \cdot 9 = 8 + 9 = 17$

$(9 + 1)(1 + 8) \rightarrow 9 \cdot 8 + 1 \cdot 1 = 72 + 1 = 73$

$(3 + 1)(3 + 8) \rightarrow 3 \cdot 8 + 1 \cdot 3 = 24 + 3 = 27$

$(1 + 2)(9 + 4) \rightarrow 1 \cdot 4 + 2 \cdot 9 = 4 + 18 = 22$

$\mathbf{(3 + 2)(3 + 4) \rightarrow 3 \cdot 4 + 2 \cdot 3 = 12 + 6 = 18}$ ☺

Step 4. Use the factors that work to write the trinomial in factored form. $9x^2 + 18x + 8 = (3x + 2)(3x + 4)$

Step 5. Check by multiplying.

$$(3x + 2)(3x + 4) = 9x^2 + 12x + 6x + 8$$
$$= 9x^2 + 18x + 8$$

With practice, you'll get faster at just listing the factors, doing the products and sums in your head, and discovering the right combination. And watch your signs!

Another method for factoring trinomials $\mathbf{ax^2 + bx + c}$ is to **factor by grouping.**

Grouping is a step-by-step method for factoring a trinomial. If you have trouble with the trial-and-error method, this method may work better for you.

Let's work with the following: $\mathbf{6x^2 - 15x + 6}$

Step 1. Multiply the coefficient on the first term and the final constant $6 \cdot 6 = 36$

Step 2. Factor this product.

$$36 = \begin{array}{l} \pm 2 \cdot \pm 18 \\ \pm 3 \cdot \pm 12 \\ \pm 4 \cdot \pm 9 \\ \pm 6 \cdot \pm 6 \end{array}$$

Step 3. Find the pair of factors that add up to the middle term of the trinomial (-15). $-3 - 12 = -15$

Step 4. Rewrite the middle term of the trinomial as the sum of these two factors. $6x^2 \mathbf{- 15x} + 6 = 6x^2 \mathbf{- 12x - 3x} + 6$

sign change!

Step 5. Group the first two and the last two terms and take out common factors. Watch your signs!

$$(6x^2 - 12x) - (3x - 6)$$
$$= 6x(x - 2) - 3(x - 2)$$

Step 6. Take out the common binomial to finish the factoring. Check your work by multiplying. $(6x - 3)(x - 2)$

If Step 5 doesn't produce a common factor, try switching the middle two terms to create a different grouping.

Factoring a trinomial by grouping takes more time. However, the method is also useful to check a trinomial for factorability, that is, is this trinomial prime and can't be factored?

Here's how to **check for "primeness"** of the trinomial $2x^2 - 5x - 20$.

Step 1. Multiply the coefficient on the first term by the last term.

$2 \cdot (-20) = -40$

Step 2. Factor the product (-40) and sum the factors of each pair. Does any sum equal the middle term of –5?

$2 + (-20) = -18$	$-2 + 20= 18$
$4 + (-10) = -6$	$-4 + 10 = 6$
$8 + (-5) = 3$	$-8 + 5 = -3$
$1 + (-40) = -39$	$-1 + 40 = 39$

No – we can't get the middle term, so this trinomial is not factorable.

Sample Problems

Find three factorizations for the monomial

1. $25x^4$ 2. $–45x^3$ 3. $8x^6$

Factor; check by multiplying

4. $8x^5 + 6x^2$ 5. $21x^5y^2 + 27x^3y^3 – 18x^2y$

6. $1.2x^7 + 2.4x^5 - 0.6x^3 + 1.8x^2$

Factor:

7. $x^2(y - 3) + 5(y – 3)$

8. $4y(2z + 1) + (2z + 1)$

Factor by grouping

9. $4x^3 + x^2 + 8x + 2$

10. $14x^3 – 21x^2 – 12x + 18$

11. $3x^2y^2 + 6x^2 – y^2 - 2$

Factor completely

12. $x^3 + x^2 – 56x$

13. $x^2 – 9x + 18$

14. $5x^5 + 35x^4 - 90x^3$

15. $5x^2 + 27x + 10$

16. $9a^2 – 21a + 12$

17. $16x^2 – 6xy – 27y^2$

Answers

1. $5x \cdot 5x^3$, $5x^2 \cdot 5x^2$, $25x^3 \cdot x$, $25x^2 \cdot x^2$, $25x \cdot x^3$
2. $-9 \cdot 5x^3$ or $9 \cdot (-5x^3)$, $-9x \cdot 5x^2$, $-9x^2 \cdot x$, $-45 \cdot x^3$, $-45x \cdot x^2$, $-45x^2 \cdot x$
3. $2 \cdot 4x^6$, $2x \cdot 4x^5$, $2x^2 \cdot 4x^4$, $2x^3 \cdot 4x^3$, $2x^4 \cdot 4x^2$, $2x^5 \cdot 4x$, $2x^6 \cdot 4$, $8 \cdot x^6$, $8x \cdot x^5$, etc
4. $2x^2(4x^3 + 3)$
5. $3x^2y(7x^3y + 9xy^2 - 6)$
6. $0.6x^2(2x^5 + 4x^3 - x + 3)$
7. $(x^2 + 5)(y - 3)$
8. $(4y + 1)(2z + 1)$
9. $(x^2 + 2)(4x + 1)$
10. $(7x^2 - 6)(2x - 3)$
11. $(3x^2 - 1)(y^2 + 2)$
12. $x(x + 8)(x - 7)$
13. $(x - 6)(x - 3)$
14. $5x^3(x + 9)(x - 2)$
15. $(5x + 2)(x + 5)$
16. $3(3a - 4)(a - 1)$
17. $(8x + 9y)(2x - 3y)$

□