EXPONENTS AND THEIR PROPERTIES (4.1, 4.2)

Basic Definitions and Notation

2^3 means $2 \bullet 2 \bullet 2$ or 8 The number 3 is the **exponent**. The number 2 is the **base**.
x^4 means $x \bullet x \bullet x \bullet x$ 4 is the **exponent**; x is the **base**.

WARNING!
The base of an exponent is the symbol directly in front of the exponent.

-2^3 means $-(2 \bullet 2 \bullet 2)$ The symbol directly in front of the exponent is 2.
$(-2)^3$ means $(-2)(-2)(-2)$ The symbol directly in front of the exponent is a parenthesis. The exponent belongs to everything in the parentheses.
$3x^4$ means $3(x \bullet x \bullet x \bullet x)$
$(3x)^4$ means $(3x)(3x)(3x)(3x)$
$-x^4$ means $-(x \bullet x \bullet x \bullet x)$

x^{-3} means $\frac{1}{x^3}$ A negative exponent means the reciprocal of the base with a *positive exponent*. It does not indicate a negative number!

$-x^{-3} = -\frac{1}{x^3}$ The negative sign *in front* remains. The negative exponent produces a reciprocal.

Properties of Exponents

1. $x^a \bullet x^b = x^{a+b}$ **Product Rule:** When **multiplying** 2 quantities with the same base, keep the common base and **add the exponents.**

Example: $5^3 \bullet 5^4 = (5 \bullet 5 \bullet 5) \bullet (5 \bullet 5 \bullet 5 \bullet 5) = (5 \bullet 5 \bullet 5 \bullet 5 \bullet 5 \bullet 5 \bullet 5) = 5^7 = 5^{3+4}$

2. $\frac{x^a}{x^b} = x^{a-b}$ **(x ≠ 0)** **Quotient Rule:** When **dividing** 2 quantities with the same base, keep the common base and s**ubtract** the denominator exponent from the numerator exponent.

Example: $\frac{3^6}{3^2} = \frac{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3}{3 \cdot 3} = \frac{\cancel{3} \cdot \cancel{3} \cdot 3 \cdot 3 \cdot 3 \cdot 3}{\cancel{3} \cdot \cancel{3}} = 3 \bullet 3 \bullet 3 \bullet 3 = 3^4 = 3^{6-2}$

3. $\mathbf{x^0 = 1}$ $\mathbf{(x \neq 0)}$ **Zero Exponent**: <u>Any base</u> (except 0) raised to the zero power equals 1.

<u>Examples</u>: $5^0 = 1$ $15432^0 = 1$ $(x^3 y^{34} z)^0 = 1$

4. Power Rules

a) $\mathbf{(x^a)^b = x^{ab}}$ If an exponential expression is raised to another power, keep the base, **multiply the exponents**.

<u>Example</u>: $(3^2)^3 = (3^2)(3^2)(3^2) = (3 \bullet 3)(3 \bullet 3)(3 \bullet 3) = 3 \bullet 3 \bullet 3 \bullet 3 \bullet 3 \bullet 3 = 3^6 = 3^{(3\bullet 2)}$

Warning:

$(3^2)^3$ and $3^2 \bullet 3^3$ are *different situations with different exponent rules!* Learn to tell them apart!

$(3^2)^3 = 3^{2 \bullet 3} = 3^6$

$3^2 \bullet 3^3 = 3^{2+3} = 3^5$

b) $\mathbf{(xy)^a = x^a y^a}$ If you have a product being raised to a power, you can raise each factor in the product to that power.

<u>Example</u>: $(2x)^3 = (2x)(2x)(2x) = (2)(2)(2)(x)(x)(x) = 2^3x^3$ or $8x^3$

c) $\mathbf{\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}}$ If you have a quotient being raised to a power, you can raise both the numerator and the denominator to that power.

<u>Example</u>: $\left(\frac{2}{3}\right)^4 = \left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right) = \frac{2 \cdot 2 \cdot 2 \cdot 2}{3 \cdot 3 \cdot 3 \cdot 3} = \frac{2^4}{3^4}$

5 Negative Exponents:

a) $\mathbf{x^{-a} = \frac{1}{x^a}}$ A negative exponent means the <u>reciprocal</u> of the base with a positive exponent.

<u>Examples</u>: $(xyz)^{-5} = \frac{1}{(xyz)^5}$ $5^{-2} = \frac{1}{5^2} = \frac{1}{25}$ $x^2(3y)^{-3} = \frac{x^2}{(3y)^3} = \frac{x^2}{3^3y^3} = \frac{x^2}{27y^3}$

b) $\dfrac{\mathbf{x}^{-a}}{\mathbf{y}^{-b}} = \dfrac{\mathbf{y}^{b}}{\mathbf{x}^{a}}$

The factor in the numerator with a negative exponent moves to the denominator. The factor in the denominator with a negative exponent moves to the numerator. Both exponents become positive.

Examples: $\dfrac{\mathbf{4}^{-3}}{\mathbf{5}^{-2}} = \dfrac{\mathbf{5}^{2}}{\mathbf{4}^{3}} = \dfrac{25}{64}$

$$\frac{xy^{-5}}{z^2w^{-6}} = \left(\frac{x}{z^2}\right)\left(\frac{\mathbf{y}^{-5}}{\mathbf{w}^{-6}}\right) = \left(\frac{x}{z^2}\right)\left(\frac{\mathbf{w}^{6}}{\mathbf{y}^{5}}\right) = \frac{xw^6}{z^2y^5}$$

Warning:

This only works with <u>factors</u>. It will not work if sums or differences are involved.

$\dfrac{x^{-2}+y^{-1}}{z^{-3}}$ **<u>cannot</u> be inverted to the form** $\dfrac{z^3}{x^2+y^1}$ **!**

c) $\left(\dfrac{\mathbf{x}}{\mathbf{y}}\right)^{-a} = \left(\dfrac{\mathbf{y}}{\mathbf{x}}\right)^{a}$

A quotient with a negative exponent <u>outside</u> the parentheses may be inverted inside the parentheses, with a positive exponent on the outside.

Examples: $\left(\dfrac{\mathbf{5}}{\mathbf{3}}\right)^{-3} = \left(\dfrac{\mathbf{3}}{\mathbf{5}}\right)^{3} = \dfrac{3^3}{5^3} = \dfrac{27}{125}$

$$\left(\frac{\mathbf{x}^3\mathbf{y}}{\mathbf{z}^2}\right)^{-7} = \left(\frac{\mathbf{z}^2}{\mathbf{x}^3\mathbf{y}}\right)^{7}$$

More Examples and Using the Properties Together

Simplify the following:

1) $x^3 y^2 x^4$ — The properties of exponents apply <u>only when the bases are the same.</u> Start by grouping like bases together.

$= x^3 x^4 y^2$ — Use the product rule and add exponents.

$= x^{3+4} y^2$

$= x^7 y^2$ — Cannot be simplified further (bases are not the same).

2) $(5x^3)^2$ — Use the power rule to distribute the exponent and square each factor of the product.

$= 5^2 (x^3)^2$ — Use the power rule (raising x^3 to second power) and multiply exponents.

$= 5^2 x^6$

$= 25x^6$

Note: If your base is a number, you may multiply it out if you wish or you may leave it in exponential form. Either form is acceptable. As a rule of thumb, multiply it out if it's easy, leave it in exponential form if it's too much work.

3) $-x^2 (x^3)^4$ — Tricky - be careful! Start with the $(x^3)^4$ and simplify.

$= -x^2 \bullet x^{12}$ — Both bases are x's (the negative sign is not involved) so you can simplify further.

$= -(x^2 \bullet x^{12})$

$= -(x^{14})$

$= -x^{14}$

4) $\left(\dfrac{x^4y^3z^7}{17x^2yz^5}\right)^0$ — This "looks" like a difficult problem. Resist the urge to simplify inside the parentheses first. Look at the exponent outside those parentheses.

$= 1$ — Any base (except 0) raised to the zero power equals 1.

5) $\left(\dfrac{2x^4}{3x^3}\right)^5$ — You may raise each factor in numerator and denominator to the fifth power or you may simplify inside the parentheses first.

$$= \frac{2^5(x^4)^5}{3^5(x^3)^5} \qquad \textit{OR} \qquad = \left(\frac{2}{3} \bullet \frac{x^4}{x^3}\right)^5$$

$$= \frac{2^5 x^{20}}{3^5 x^{15}} \qquad\qquad = \left(\frac{2}{3} \bullet x^{4-3}\right)^5$$

$$= \frac{2^5}{3^5} \bullet \frac{x^{20}}{x^{15}} \qquad\qquad = \left(\frac{2}{3} \bullet x^1\right)^5$$

$$= \frac{2^5}{3^5} \bullet x^{20-15} \qquad\qquad = \left(\frac{2x}{3}\right)^5$$

$$= \frac{2^5}{3^5} \bullet x^5 = \frac{2^5x^5}{3^5} \qquad\qquad = \frac{2^5x^5}{3^5}$$

Cannot be simplified further; bases are not the same.

6) $\dfrac{x^5}{x^{-4}}$ Since the bases are the same, we don't need to use any negative exponent properties for this problem. Just use the quotient rule.

$= x^{5-(-4)}$

$= x^{5+4}$

$= x^9$

7) $\dfrac{(x-3y)^{-4}}{(xy^2)^{-7}}$ Use the negative exponent rules to invert the quotient with positive exponents.

$= \dfrac{(xy^2)^7}{(x-3y)^4}$ or $\dfrac{x^7y^{14}}{(x-3y)^4}$

Remember: $(a+b)^m \neq a^m + b^m$!
$(x-3y)^4$ cannot be further simplified.

8) $\dfrac{(5^{-1}x^{-3})^{-1}(2xy^2)^3}{(3x^{-2}y^3)^{-2}}$ Here's a jumble of positive and negative exponents all mixed up. A simple way to proceed is to first raise all the inner factors, including those with exponents, to their outer exponents.

$= \dfrac{(5^{-1})^{-1}(x^{-3})^{-1}(2^3)(x)^3(y^2)^3}{(3^{-2})(x^{-2})^{-2}(y^3)^{-2}}$ Use the power rule to **multiply exponents.**

$= \dfrac{5^1\,x^3\,2^3\,x^3\,y^6}{3^{-2}\,x^4\,y^{-6}}$ Group numbers and like bases in the numerator. Multiply numbers and use the product rule on the variables to simplify.

* $= \dfrac{40x^6y^6}{3^{-2}x^4y^{-6}}$ Separate terms with positive exponents from those with negative exponents

$= \dfrac{40x^6y^6}{x^4} \bullet \dfrac{1}{3^{-2}} \bullet \dfrac{1}{y^{-6}}$ Use the negative exponent property to create reciprocals with positive exponents

$= \dfrac{40x^6y^6}{x^4} \bullet \dfrac{3^2}{1} \bullet \dfrac{y^6}{1}$ Use the product and quotient rules on like variables.

$= 40 \bullet 9\ x^{(6-4)}\ y^{(6+6)} = 360x^2y^{12}$

* The quotient rule could have been applied at this step on the variables.

Problems

Simplify:

1) $x^4 x^9 x^2$

2) $(7y)^4 (7y)^3$

3) $5x^3 (3x^2)$

4) $\dfrac{3y^5}{2y^2}$

5) $\dfrac{x^8 y^{10} z^5}{x^4 y^{10} z}$

6) $(2xy^4)^3$

7) $\left(\dfrac{5z^5}{z^3}\right)^2$

8) $\left(\dfrac{-2x^4}{xy^2}\right)^4$

9) $\left(\dfrac{4x^3 y^2 z^5}{3xy}\right)^2$

10) $\left(\dfrac{5x^5 y^{10} z^8}{8xz^2}\right)^0$

11) $x^2 + x^5$

12) $(5m)^4 (5m)^{-3}$

13) $\left(\dfrac{a^{-1}b}{c^2}\right)^{-2}$

14) $(7x^{-6} z^4)^{-3}$

Answers

1) x^{15}

2) $7^7 y^7$ or $(7y)^7$

3) $15x^5$

4) $\dfrac{3y^3}{2}$

5) $x^4 z^4$

6) $2^3 x^3 y^{12}$ or $8x^3 y^{12}$

7) $5^2 z^4$ or $25z^4$

8) $\dfrac{2^4 x^{12}}{y^8}$ or $\dfrac{16x^{12}}{y^8}$

9) $\dfrac{4^2 x^4 y^2 z^{10}}{3^2}$ or $\dfrac{16x^4 y^2 z^{10}}{9}$

10) 1

11) $x^2 + x^5$ (Trick question. x^2 and x^5 are not like terms and cannot be added. There are no exponent properties that apply to this situation!)

12) $5m$

13) $\dfrac{a^2 c^4}{b^2}$

14) $\dfrac{x^{18}}{343z^{12}}$