15 Dividing Polynomials (4.7)

Monomial ÷ Monomial

Example 1: Divide the monomial $10x^3y^3$ by $15xy^2$.

$$\frac{10x^3y^3}{15xy^2}$$

$$= \frac{10 \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y}{15 \cdot x \cdot y \cdot y}$$ Method 1: Expand the terms

$$= \frac{\overset{2}{\cancel{10}} \cdot \cancel{x} \cdot x \cdot x \cdot \cancel{y} \cdot \cancel{y} \cdot y}{\underset{3}{\cancel{15}} \cdot \cancel{x} \cdot \cancel{y} \cdot \cancel{y}}$$ Cancel terms and reduce as you would any fraction

$$= \frac{2x^2y}{3}$$

$$= \frac{\overset{2}{\cancel{10}} \cdot x^{(3-1)} \cdot y^{(3-2)}}{\underset{3}{\cancel{15}}}$$ Method 2: Use the quotient rule to subtract exponents

$$= \frac{2x^2y}{3}$$

Polynomial ÷ Monomial

Use the property of fractions $\frac{A+B}{C} = \frac{A}{C} + \frac{B}{C}$ to rewrite as monomials divided by a monomial. Then simplify as before.

Example 2: $\frac{12x^3y^3 - 3x^2y^2 + 6xy^3 - 6xy^2}{6xy^2}$

$$= \frac{12x^3y^3}{6xy^2} - \frac{3x^2y^2}{6xy^2} + \frac{6xy^3}{6xy^2} - \frac{6xy^2}{6xy^2}$$

$$= \frac{12 \cdot x^3 \cdot y^3}{6 \cdot x \cdot y^2} - \frac{3 \cdot x^2 \cdot \cancel{y^2}}{6 \cdot x \cdot \cancel{y^2}} + \frac{\cancel{6 \cdot x} \cdot y^3}{\cancel{6 \cdot x} \cdot y^2} - \frac{\cancel{6 \cdot x \cdot y^2}}{\cancel{6 \cdot x \cdot y^2}}$$

$$= 2x^{(3-1)}y^{(3-2)} - \frac{x^{(2-1)}}{2} + y^{(3-2)} - 1$$

$$= 2x^2y - \frac{x}{2} + y - 1$$

Polynomial ÷ Polynomial

These must be done by "long division".

Example 3: Divide $4x^3 - 6x^2 + 2x + 5$ by $2x + 1$

Step 1: Set up your dividend and divisor (both in descending powers, for a long division)

$$2x + 1 \overline{\big)\, 4x^3 - 6x^2 + 2x + 5}$$

Step 2: Look at the first term in each and decide what you have to multiply 2x by to get $4x^3$. Write that term on top.

$$\begin{array}{r l} & \mathbf{2x^2} \\ \mathbf{2x} + 1 & \overline{\big)\, \mathbf{4x^3} - 6x^2 + 2x + 5} \end{array}$$

Step 3: Multiply the 2x + 1 by the $2x^2$. Write these terms under the first terms in the dividend.

$$\begin{array}{r l} & 2x^2 \\ 2x + 1 & \overline{\big)\, 4x^3 - 6x^2 + 2x + 5} \\ & \;\; 4x^3 + 2x^2 \end{array}$$

Step 4: Subtract the bottom terms from the top terms (or change the signs of the bottom terms and add them to the terms above).
Note: If the first terms do not add to zero, you put the wrong number on top! Try again.

$$\begin{array}{r l} & 2x^2 \\ 2x + 1 & \overline{\big)\, 4x^3 - 6x^2 + 2x + 5} \\ & - \underline{(4x^3 + 2x^2)} \\ & \quad 0 \; - 8x^2 \end{array}$$

Step 5: Bring down the next term (or terms) from the dividend so you have the same number of terms as the divisor.

$$\begin{array}{r l} 2x + 1 & \overline{\big)\, 4x^3 - 6x^2 + 2x + 5} \\ & - \underline{4x^3 - 2x^2} \\ & \qquad\; - 8x^2 + 2x \end{array}$$

Step 6: Start the whole process over (from Step 2). (Decide what you have to multiply 2x by to get $-8x^2$. Write that term on top.)

$$\begin{array}{r l} & 2x^2 - \mathbf{4x} \\ \mathbf{2x} + 1 & \overline{\big)\, 4x^3 - 6x^2 + 2x + 5} \\ & - \underline{4x^3 - 2x^2} \\ & \qquad\; \mathbf{- 8x^2} + 2x \end{array}$$

Step 3 again: Multiply the 2x + 1 by – 4x.

$$\begin{array}{r l} & 2x^2 - 4x \\ 2x + 1 & \overline{\big)\, 4x^3 - 6x^2 + 2x + 5} \\ & - \underline{4x^3 - 2x^2} \\ & \qquad\; - 8x^2 + 2x \\ & \qquad\; - 8x^2 - 4x \end{array}$$

Step 4 again: <u>Subtract</u> the bottom terms from the terms above (or change the signs and add).

$$\begin{array}{r}
2x^2 \; - \; 4x \qquad\qquad \\
2x+1 \;\overline{\big)\, 4x^3 - 6x^2 + 2x + 5} \\
-\;\underline{4x^3 \; - 2x^2} \qquad\qquad \\
-8x^2 \; + 2x \quad \\
-\;\underline{(-8x^2 \; - 4x)} \quad \\
0 \quad + 6x \quad
\end{array}$$

Step 5 again: Bring down the next term.

$$\begin{array}{r}
2x^2 \; - \; 4x \qquad\qquad \\
2x+1 \;\overline{\big)\, 4x^3 - 6x^2 + 2x + 5} \\
-\;\underline{4x^3 \; - 2x^2} \qquad\qquad \\
-8x^2 \; + 2x \quad \\
+\underline{8x^2 \; + 4x} \quad \\
6x + 5
\end{array}$$

Step 6 again: Start the process again.

$$\begin{array}{r}
2x^2 \; - \; 4x \; + \mathbf{3} \qquad \\
\mathbf{2x}+1 \;\overline{\big)\, 4x^3 - 6x^2 + 2x + 5} \\
-\;\underline{4x^3 \; - 2x^2} \qquad\qquad \\
-8x^2 \; + 2x \quad \\
+\underline{8x^2 \; + 4x} \quad \\
\mathbf{6x} + 5
\end{array}$$

Step 3 again:

$$\begin{array}{r}
2x^2 \; - \; 4x \; + 3 \qquad \\
2x+1 \;\overline{\big)\, 4x^3 - 6x^2 + 2x + 5} \\
-\;\underline{4x^3 \; - 2x^2} \qquad\qquad \\
-8x^2 \; + 2x \quad \\
+\underline{8x^2 \; + 4x} \quad \\
6x + 5 \\
6x + 3
\end{array}$$

Step 4 again:

$$\begin{array}{r}
2x^2 \; - \; 4x \; + 3 \qquad \\
2x+1 \;\overline{\big)\, 4x^3 - 6x^2 + 2x + 5} \\
-\;\underline{4x^3 \; - 2x^2} \qquad\qquad \\
-8x^2 \; + 2x \quad \\
+\underline{8x^2 \; + 4x} \quad \\
6x + 5 \\
\underline{-(6x + 3)} \\
0 \; + 2
\end{array}$$

There are no more terms to bring down and the degree of the divisor is now larger than the degree of our remainder. We are done with the calculations!

Final Step: Write the remainder as part of the quotient on top by adding the remainder divided by the divisor:

$$
\begin{array}{r l}
 & 2x^2 - 4x + 3 + \dfrac{2}{2x+1} \\
2x+1 & \overline{)\,4x^3 - 6x^2 + 2x + 5} \\
 & -\ \underline{4x^3 - 2x^2} \\
 & -8x^2 + 2x \\
 & \ \ \underline{8x^2 + 4x} \\
 & \qquad\quad 6x + 5 \\
 & \qquad\quad \underline{-6x - 3} \\
 & \qquad\qquad\quad 2
\end{array}
$$

A VERY IMPORTANT NOTE!

In these problems, it is very important (in the "change signs and add" step) that like terms line up directly underneath each other. When you first write the problem down and set it up for the long division process, make sure you have a column for every power of the variable between your first and last term. For example, if the dividend is $x^5 - 2x^3 + 7$, I would write $x^5+0-2x^3+0+0+7$. This way I reserve a column for terms with x^4, x^2, and x. Do the same for your divisor. If my divisor is $2x^2 - 1$, I would write $2x^2 + 0 - 1$ (reserving a column for terms with x in them). If you do this, everything lines up perfectly and you won't get confused. Just be careful when carrying out the arithmetic with the zeros. Remember, adding or subtracting 0 has no effect; multiplying something by 0 gives 0.

Example 4 (a hard one): Divide $x^4 - x^3 - 5x + 3$ by $x^2 - 2$

Set up as long division, reserving columns with zeros. What do I multiply x^2 by to get x^4? Write x^2 on top. Multiply $x^2 + 0 - 2$ by x^2.

$$
\begin{array}{r l}
 & \mathbf{x^2} \\
\mathbf{x^2} + 0 - 2 & \overline{)\,\mathbf{x^4} - x^3 + 0 - 5x + 3} \\
 & x^4 + 0 - 2x^2
\end{array}
$$

Change signs and add. Bring down the next term.

$$
\begin{array}{r l}
 & x^2 \\
x^2 + 0 - 2 & \overline{)\,x^4 - x^3 + 0 - 5x + 3} \\
 & \underline{-x^4 - 0 + 2x^2} \\
 & \qquad -x^3 + 2x^2 - 5x
\end{array}
$$

Do it again. What do I multiply x^2 by to get $-x^3$? Write $-x$ on top. Multiply $x^2 + 0 - 2$ by $-x$.

$$
\begin{array}{r l}
 & x^2 \mathbf{- x} \\
\mathbf{x^2} + 0 - 2 & \overline{)\,x^4 - x^3 + 0 - 5x + 3} \\
 & \underline{-x^4 - 0 + 2x^2} \\
 & \qquad \mathbf{-x^3} + 2x^2 - 5x \\
 & \qquad \underline{-x^3 - 0 + 2x}
\end{array}
$$

Change signs and add. Bring down the next term.

$$
\begin{array}{r}
x^2 - x \\
x^2+0-2\overline{)\,x^4 - x^3 + 0 - 5x + 3} \\
\underline{-x^4 - 0 + 2x^2} \\
-x^3 + 2x^2 - 5x \\
+\underline{x^3 + 0 - 2x} \\
2x^2 - 7x + 3
\end{array}
$$

Do it again. What do I multiply x^2 by to get $2x^2$? Write +2 on top. Multiply $x^2 + 0 - 2$ by +2.

$$
\begin{array}{r}
x^2 - x + \mathbf{2} \\
\mathbf{x^2}+0-2\overline{)\,x^4 - x^3 + 0 - 5x + 3} \\
\underline{-x^4 - 0 + 2x^2} \\
-x^3 + 2x^2 - 5x \\
+\underline{x^3 + 0 - 2x} \\
\mathbf{2x^2} - 7x + 3 \\
2x^2 - 0 - 4
\end{array}
$$

Change signs and add. There are no more terms to bring down and the degree of the divisor is larger than the degree of the remainder. We are done. Write the remainder as part of the quotient by adding the remainder divided by the divisor.

$$
\begin{array}{r}
x^2 - x + 2 + \dfrac{-7x+7}{x^2-2} \\
x^2+0-2\overline{)\,x^4 - x^3 + 0 - 5x + 3} \\
\underline{-x^4 - 0 + 2x^2} \\
-x^3 + 2x^2 - 5x \\
+\underline{x^3 + 0 - 2x} \\
2x^2 - 7x + 3 \\
\underline{-2x^2 + 0 + 4} \\
-7x + 7
\end{array}
$$

Practice Problems

1. $(24x^5 + 12x^3 - 6x^2) \div (3x^2)$

2. $\dfrac{8x^4 - 4x^3 + 12x}{-8x}$

3. $(5x^2 - 10x + 15) \div (x-1)$

4. $\dfrac{2x^4 + x^3 - 4x^2 + 3x - 2}{2x^2 - x + 1}$

5. Divide $15x^3 - 7x^2 - 11x + 4$ by $3x - 2$.

6. $(x^4 + 8x^2 - 12) \div (x - 2)$

7. $\dfrac{x^5 - x + 1}{x^2 + 1}$

Answers

1. $8x^3 + 4x - 2$

2. $-x^3 + \frac{x^2}{2} - \frac{3}{2}$

3. $5x - 5 + \frac{10}{x-1}$

4. $x^2 + x - 2$

5. $5x^2 + x - 3 + \frac{-2}{3x-2}$

6. $x^3 + 2x^2 + 12x + 24 + \frac{36}{x-2}$

7. $x^3 - x + \frac{1}{x^2+1}$