27 COMPLETING THE SQUARE

There are several ways to solve quadratic equations, one of which is called "completing the square". This method is frequently used to get equations for circles, ellipses, parabolas, and hyperbolas into standard forms to quickly find where the centers or vertices of these conic sections are located.

Completing the square means *creating* a perfect square of a binomial in order to exploit the *square root property* of solving quadratic equations. For example, if $x^2 = 25$, then $x = \pm\sqrt{25} = \pm 5$. Similarly, if $(x - 1)^2 = 25$, then $x - 1 = \pm 5$ or $x = 1 \pm 5$.
If you have a trinomial that cannot be factored, you can complete the square to solve it.

We'll start with equations in one variable, in the quadratic form $\boldsymbol{ax^2 + bx + c = 0}$.

When a = 1:

Step 1. Move the constant *c* to the other side: $\boldsymbol{x^2 + bx = -c.}$

Step 2. Divide the coefficient *b* by 2, square it, and add it to both sides:

$$\boldsymbol{x^2 + bx + (b/2)^2 = -c + (b/2)^2}$$

Step 3. You've now created a trinomial that is the perfect square of a binomial.
Rewrite: $\boldsymbol{(x + b/2)^2 = -c + (b/2)^2}$

Note that the number you get when you divide *b* by 2 is what goes into the binomial!

Step 4. Take the square root of both sides: $\boldsymbol{x + b/2 = \pm\sqrt{-c + (b/2)^2}}$

Step 5. Solve for x: $\boldsymbol{x = -b/2 \pm \sqrt{-c + (b/2)^2}}$

<u>Example 1</u> Solve $x^2 + 6x - 11 = 0$

1. Move the constant to the other side: $x^2 + 6x = 11$

2. Divide 6 by 2 (= 3), square it and add it to both sides: $x^2 + 6x + 3^2 = 11 + 3^2$
$x^2 + 6x + 9 = 11 + 9 = 20$

3. Rewrite (3 goes into the binomial) $(x + 3)^2 = 20$

4. Take the square root: $x + 3 = \pm\sqrt{20} = \pm 2\sqrt{5}$

5. Solve for x: $x = -3 \pm 2\sqrt{5}$

solution set: $\{-3 + 2\sqrt{5}, -3 - 2\sqrt{5}\}$

Example 2 Solve $x^2 - \frac{1}{2}x - 5 = 0$

1. Move the constant to the other side: $x^2 - \frac{1}{2}x = 5$

2. Divide $-\frac{1}{2}$ by 2 ($= -\frac{1}{4}$), square and add to both sides:

$$x^2 - \frac{1}{2}x + (-\tfrac{1}{4})^2 = 5 + (-\tfrac{1}{4})^2$$
$$x^2 - \frac{1}{2}x + 1/16 = 5 + 1/16$$

3. Rewrite ($-\frac{1}{4}$ goes into the binomial): $(x - \frac{1}{4})^2 = 81/16$

4. Take the square root: $x - \frac{1}{4} = \pm 9/4$

5. Solve for x: $$x = \frac{1}{4} \pm \frac{9}{4} = \frac{1 \pm 9}{4}$$

solution set: $\left\{\frac{5}{2}, -2\right\}$

Example 3 Solve $x^2 - 3x + 5 = 0$

1. Move the constant to the other side: $x^2 - 3x = -5$

2. Divide -3 by 2 ($= -3/2$), square and add to both sides:

$$x^2 - 3x + (-3/2)^2 = -5 + (-3/2)^2$$
$$x^2 - 3x + 9/4 = -5 + 9/4 = -11/4$$

3. Rewrite ($-3/2$ goes into the binomial): $(x - 3/2)^2 = -11/4$

4. Take the square root (take an *i* out of the radical): $$x - 3/2 = \pm\sqrt{\frac{-11}{4}} = \frac{\pm i\sqrt{11}}{2}$$

5. Solve for x: $$x = \frac{3}{2} \pm \frac{i\sqrt{11}}{2} = \frac{3 \pm i\sqrt{11}}{2}$$

solution set: $\left\{\frac{3 + i\sqrt{11}}{2}, \frac{3 - i\sqrt{11}}{2}\right\}$

When a ≠ 1:

If there is a coefficient on the x^2 term other than 1, divide all terms by that coefficient.

Example 4 Solve $2x^2 - 16x + 25 = 0$

1. Divide everything by 2: $x^2 - 8x + 25/2 = 0$

2. Move the constant to the other side: $x^2 - 8x = -25/2$

3. Divide -8 by 2 (= - 4), square and add to both sides: $x^2 - 8x + (-4)^2 = -25/2 + (-4)^2$

$x^2 - 8x + 16 = -25/2 + 16 = 7/2$

4. Rewrite (- 4 goes into the binomial): $(x - 4)^2 = 7/2$

5. Take the square root: $x - 4 = \pm\sqrt{\frac{7}{2}} = \pm\frac{\sqrt{14}}{2}$

6. Solve for x: $x = 4 \pm \frac{\sqrt{14}}{2}$

solution set: $\left\{4+\frac{\sqrt{14}}{2}, 4-\frac{\sqrt{14}}{2}\right\}$

Example 5 Solve $25x^2 - 20x - 1 = 0$

1. Divide everything by 25: $x^2 - \frac{4}{5}x - \frac{1}{25} = 0$

2. Move the constant to the other side: $x^2 - \frac{4}{5}x = \frac{1}{25}$

3. Divide -4/5 by 2 (= -2/5), square and add to both sides: $x^2 - \frac{4}{5}x + \left(-\frac{2}{5}\right)^2 = \frac{1}{25} + \left(-\frac{2}{5}\right)^2$

$x^2 - \frac{4}{5}x + \frac{4}{25} = \frac{1}{25} + \frac{4}{25} = \frac{1}{5}$

4. Rewrite (-2/5 goes into the binomial): $(x - 2/5)^2 = 1/5$

5. Take the square root: $x - 2/5 = \pm\sqrt{\frac{1}{5}} = \pm\frac{\sqrt{5}}{5}$

6. Solve for x: $x = \frac{2}{5} \pm \frac{\sqrt{5}}{5} = \frac{2 \pm \sqrt{5}}{5}$

solution set: $\left\{\frac{2+\sqrt{5}}{5}, \frac{2-\sqrt{5}}{5}\right\}$

Sometimes an equation needs simplifying first.

Example 6 Solve $(x + 4)(x - 1) = 5$

1. Simplify by multiplying and subtracting: $x^2 + 3x - 4 = 5$
$x^2 + 3x - 9 = 0$

Now, proceed as before. $\left(\text{solution set}: \left\{\frac{-3+3\sqrt{5}}{2}, \frac{-3-3\sqrt{5}}{2}\right\}\right)$

Applications to Conic Section Equations

Standard equations for conic sections are very useful because they immediately tell you where the center or vertex is and contain information about the orientation and shape of the graph.

Parabola

A standard equation for a vertical parabola is $(x - h)^2 = 4p(y - k)$, where (h, k) are the coordinates for the vertex and p is the distance from the vertex to both the focus and the directrix. For a parabola, you complete the square only for one variable – the one that is squared to start with.

Find the standard equation for the parabola given by: $x^2 + 5x - 4y - 1 = 0$.

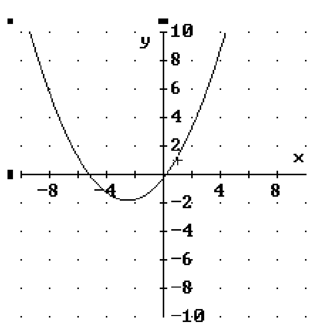
1. Move everything but the x terms to the other side: $x^2 + 5x = 4y + 1$

2. Divide 5 by 2 (= 5/2), square and add to both sides: $x^2 + 5x + (5/2)^2 = 4y + 1 + (5/2)^2$
$x^2 + 5x + 25/4 = 4y + 29/4$

3. Rewrite (5/2 goes into the binomial): $(x + 5/2)^2 = 4y + 29/4$

4. Do not take the square root!
Factor a 4 out of the right side. $(x + 5/2)^2 = 4(y + 29/16)$

Now it's in standard form: the vertex is at (-5/2, -29/16); 4p = 4, so p = 1; because it's the x variable that is squared, this is a vertical parabola; and, because 4p is a positive number, it opens upward.



Ellipse

A standard equation for an ellipse with its major axis parallel to the x-axis is:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

where *(h, k)* are the coordinates of the center, *a* is the distance from the center to the major vertices, and *b* is the distance from the center to the minor vertices ($a > b$). Ellipses and hyperbolas require completing the squares for both variables.

Find the standard equation for the ellipse given by: $9x^2 + 16y^2 + 36x - 16y - 104 = 0$.

1. Rearrange the terms to get like variables next to each other and move the constant to the right side: $9x^2 + 36x + 16y^2 - 16y = 104$

2. Factor out the coefficients in front of the x^2 and the y^2 terms, in 2 separate groups: $9(x^2 + 4x) + 16(y^2 - y) = 104$

3. Complete the squares within the parentheses of each group: $9(x^2 + 4x + 4) + 16(y^2 - y + \frac{1}{4})$

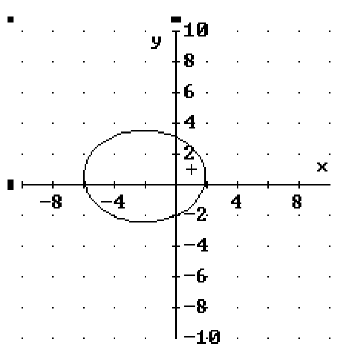
4. Now we have to compensate for the numbers we added by adding the same amount to the right side of the equation. But what did we really add? For the *x*-trinomial, we added 4 inside the parentheses, but really added a total of 9 *times* 4, or 36. For the *y*-trinomial, we added 16 *times* ¼ , or 4.

$$9(x^2 + 4x + 4) + 16(y^2 - y + \tfrac{1}{4}) = 104 + 36 + 4$$

5. Rewrite with squared binomials: $9(x + 2)^2 + 16(y - \frac{1}{2})^2 = 144$

6. The right side needs to equal 1 – divide by 144 and simplify: $$\frac{(x+2)^2}{16} + \frac{(y-1/2)^2}{9} = 1$$

Now it's in standard form: the center is at (-2, ½); the distance *a* is 4 and the distance *b* is 3; and the major axis is horizontal because the larger number is under the *x* term.



Circle

A circle is a special case of an ellipse, in which a = b.

The standard equation of a circle is: $(x-h)^2 + (y-k)^2 = r^2$, where (h, k) are the coordinates of the center and r is the radius of the circle.

<u>Find the standard equation for the circle given by:</u> $x^2 + y^2 - 6x - 4y + 8 = 0$.

1. Rearrange the terms to get like variables next to each other and move the constant to the right side: $x^2 - 6x + y^2 - 4y = -8$

2. Group the x and y terms: $(x^2 - 6x) + (y^2 - 4y) = -8$

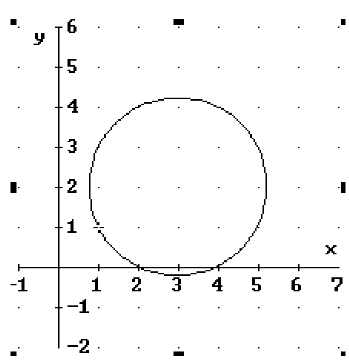
3. Complete the squares <u>within the parentheses</u> of each group: $(x^2 - 6x + 9) + (y^2 - 4y + 4)$

4. Now we have to compensate for the numbers we added by adding the same amount to the right side of the equation.

$$(x^2 - 6x + 9) + (y^2 - 4y + 4) = -8 + 9 + 4$$

5. Rewrite with squared binomials: $(x-3)^2 + (y-2)^2 = 5$

No need to get 1 on the right side – it's now in standard form. The center is at (3, 2) and the radius is $\sqrt{5}$.



Hyperbola

A standard equation for a hyperbola with its transverse axis parallel to the x-axis is:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

where *(h, k)* are the coordinates of the center, *a* is the distance from the center to the vertices, and *b* is a second number that defines, with *a*, the slant asymptotes.

<u>Find the standard equation for the hyperbola given by:</u> $4x^2 - 25y^2 + 16x + 50y - 109 = 0$

1. Rearrange the terms to get like variables next to each other and move the constant to the right side: $4x^2 + 16x - 25y^2 + 50y = 109$

2. Factor out the coefficients in front of the x^2 and the y^2 terms, <u>in 2 separate groups</u>: $4(x^2 + 4x) - 25(y^2 - 2y) = 109$

3. Complete the squares <u>within the parentheses</u> of each group: $4(x^2 + 4x + 4) - 25(y^2 - 2y + 1)$

4. Now we have to compensate for the numbers we added by adding the same amount to the right side of the equation.
We added 4 times 4, or 16, for the *x*-trinomial and
-25 times 1, or -25, for the *y*-trinomial.

$$4(x^2 + 4x + 4) - 25(y^2 - 2y + 1) = 109 + 16 - 25$$

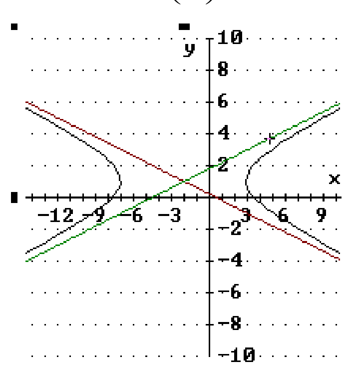
5. Rewrite with squared binomials: $4(x + 2)^2 - 25(y - 1)^2 = 100$

6. Divide by 100 and simplify: $\frac{(x+2)^2}{25} - \frac{(y-1)^2}{4} = 1$

Now it's in standard form: the center is at (-2, 1); the distance *a* is 5 and the distance *b* is 2; the major axis is horizontal because the positive (leading) term is the *x* term.

Slant asymptotes are given by: $y = k \pm \left(\frac{b}{a}\right)(x-h) \rightarrow y = 1 \pm \left(\frac{2}{5}\right)(x+2)$

or $y = \frac{2}{5}x + \frac{9}{5}$ and $y = -\frac{2}{5}x + \frac{1}{5}$



Calculus – Integrals Involving Inverse Trigonometric Functions

Two types of integrands may need completing the square to get the functions into the right format for integrating:

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\frac{u}{a} + C \qquad \text{and} \qquad \int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a} + C$$

<u>Example</u> Evaluate the integral $\displaystyle\int_{-2}^{2} \frac{dx}{x^2+4x+13}$

1. Within the denominator, group the *x* terms, complete the square, and compensate by *subtracting* the same amount that was added:

$$(x^2 + 4x + 4) + 13 - 4 = (x + 2)^2 + 9 \qquad \rightarrow \qquad \int_{-2}^{2} \frac{dx}{(x+2)^2+3^2}$$

$u = (x + 2)$, $du = dx$, and $a = 3$

2. Integrate according to the formula: $\displaystyle\int_{-2}^{2} \frac{dx}{(x+2)^2+3^2} = \frac{1}{3}\arctan\frac{x+2}{3}\Bigg|_{-2}^{2} \approx .3091$

<u>Example</u> Evaluate the integral $\displaystyle\int \frac{2}{\sqrt{-x^2+4x}}\,dx$

1. Under the radical, group the *x* terms, factor out -1, complete the square, and compensate for the amount that was added:

$$-(x^2 - 4x + 4) + 4 = 2^2 - (x - 2)^2$$

In this case, the 4 in the parentheses is actually negative, so 4 must be *added* in order to compensate.

$$\int \frac{2}{\sqrt{-x^2+4x}}\,dx = 2\int \frac{dx}{\sqrt{2^2-(x-2)^2}} \qquad a = 2,\ u = (x - 2),\ du = dx$$

2. Integrate according to the formula: $\displaystyle 2\int \frac{dx}{\sqrt{2^2-(x-2)^2}} = 2\arcsin\frac{x-2}{2} + C$