26 ABSOLUTE VALUE EQUATIONS AND INEQUALITIES

The absolute value of a number is the *magnitude* of the number without regard to the *sign* of the number. Absolute value is indicated by vertical lines and is always non-negative (positive or zero).

The statements $|5| = 5$ and $|-5| = 5$ are read " the absolute value of 5 is 5" and "the absolute value of negative 5 is 5".

Absolute value equations are of the form $|x| = a$, where a is any non-negative number. Since x could be either positive or negative, to solve the equation we split it into two equations joined with an **OR** connector.

$|x| = 9 \quad \rightarrow \quad \mathbf{x = 9} \quad \textbf{OR} \quad \mathbf{x = -9}$

Either 9 or -9 will satisfy the equation, since the absolute values of both equal 9.

In general, drop the absolute value sign and keep the right side the same for the first equation; then drop the absolute value sign and make the right side negative for the second equation. Do not otherwise change the left side!

$|x - 3| = 10 \quad \rightarrow \quad x - 3 = 10 \quad \textbf{OR} \quad x - 3 = -10$

$\mathbf{x = 13} \quad \textbf{OR} \quad \mathbf{x = -7}$

Check your answers: $|13 - 3| = |10| = 10$
$|-7 - 3| = |-10| = 10$

Solve: $|x - 3| = x + 7$

$x - 3 = x + 7$ **OR** $x - 3 = -(x + 7) = -x - 7$
$-3 = 7$ **OR** $2x = -4 \rightarrow \mathbf{x = -2}$
no solution check: $|-2 - 3| = -2 + 7$
$|-5| = 5$
$5 = 5$

Solve: $2|x + 2| - 3 = 5$

First step: get the absolute value term by itself on one side of the equation

$2|x + 2| - 3 = 5 \quad \rightarrow \quad 2|x + 2| = 8 \quad \rightarrow \quad |x + 2| = 4$

Now we can do the split: $x + 2 = 4$ **OR** $x + 2 = -4$
$\mathbf{x = 2}$ **OR** $\mathbf{x = -6}$

Check:	2| 2 + 2 | - 3 = 5	**OR**	2| -6 + 2 | - 3 = 5
	2| 4 | - 3 = 5	**OR**	2| -4 | - 3 = 5
	2 • 4 – 3 = 5	**OR**	2 • 4 – 3 = 5
	5 = 5		5 = 5

What if you have absolute value terms on both sides? Just ignore one of them (like the one on the right side) and solve as usual.

So, the equation $|x-3| = |x+7|$ would be solved exactly like the equation $|x-3| = x+7$: drop the absolute value sign on the right side and proceed as before.
Note: This method only works with equalities!

Trick question: solve $|x^2 - x + 7| = -9$.
Smart answer: No solution. The absolute value can never be negative!

Absolute value inequalities come in two varieties, "less than" and "greater than". The sign of the inequality is crucial to setting up the problem correctly. However, do not decide which way the sign is going until you isolate the (positive) absolute value term.

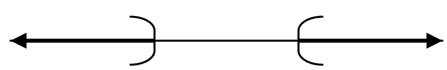
1. "Greater than" inequalities set up much like the equalities in that you get two inequalities joined with the **OR** connector. However, the critical difference comes in reversing the inequality sign when you make the right-hand side negative.

$|x| > 10 \quad \rightarrow \quad x > 10$ **OR** $x < -10$

Notice that in the second inequality, the inequality sign is reversed to go with the -10.

The solution, in interval notation, is: $(-\infty, -10) \cup (10, \infty)$. An **OR** statement translates to a union of the two separate solution intervals, and a union means "include everything".

On a number line it would look like:



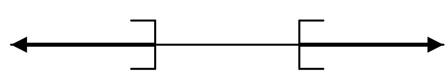
Solve: $|x+5| + 6 \geq 27$

First step: get the absolute value term by itself on one side of the equation

$|x+5| + 6 \geq 27 \quad \rightarrow \quad |x+5| \geq 21$

$x + 5 \geq 21$	**OR**	$x + 5 \leq -21$
$\mathbf{x \geq 16}$	**OR**	$\mathbf{x \leq -26}$

The solution is: $(-\infty, -26] \cup [16, \infty)$



Remember we use the square bracket when we can include the value ("less than or equal to" or "greater than or equal to"), and the parenthesis when we can't include the value (which always applies to infinity).

Trick question: solve $|4x + 1| - 2 > -5$
First, get the absolute value term by itself: $|4x + 1| > -3$
Smart answer: Always true: $(-\infty, \infty)$. The absolute value is always greater than a negative number!

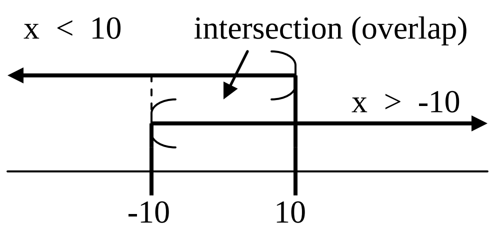
1. "Less than" inequalities produce two inequalities joined by an **AND** statement. **AND** means the *intersection* of the two solutions – the total solution must satisfy the conditions of <u>both</u> solutions <u>at the same time</u>. So, don't include everything, just the overlap. This results in a closed interval, vs. the open-ended intervals seen in "greater than" inequalities. Again, reverse the inequality sign when you make the right side negative.

$|x| < 10 \quad \rightarrow \quad x < 10$ **AND** $x > -10$

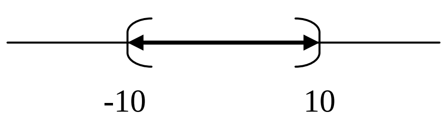
x must be <u>both</u> less than 10 <u>and</u> greater than -10.

This can be written as the compound inequality $-10 < x < 10$ or the interval $(-10, 10)$

To picture how this works, let's graph the individual solutions:



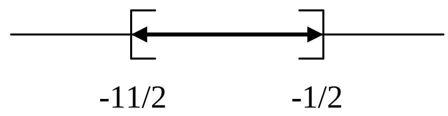
Because we have to be in both solutions at the same time, the solution to the problem is the <u>intersection</u> of the individual solutions:



Solve: $|-2x - 6| \leq 5$

$-2x - 6 \leq 5$	**AND**	$-2x - 6 \geq -5$
$-2x \leq 11$	**AND**	$-2x \geq 1$
$x \geq -11/2$	**AND**	$x \leq -1/2$

(remember to reverse the sign when dividing or multiplying by a negative number) solution: $-\frac{11}{2} \le x \le -\frac{1}{2}$ or [-11/2, -1/2]



"Less than" inequalities can also be solved by setting up the compound inequality first, then solving the whole thing at the same time.

Solve: $-2|-t + 6| \ge -14$ Is this an **AND** or an **OR**?

Isolate the absolute value term first: $\frac{-2|-t+6|}{-2} \le \frac{-14}{-2}$ = $|-t + 6| \le 7$ (reverse the sign)

Set up the compound inequality: $-7 \le -t + 6 \le 7$
Subtract 6 from the left, middle, and right: $-13 \le -t \le 1$
Multiply by negative 1, all sections (switch signs!): $13 \ge t \ge -1$
Put in the more usual order: $-1 \le t \le 13$ or [-1, 13]

Solve: You need to cut a board to a length of 13 inches. If you can tolerate no more than a 2% relative error, what would be the boundaries of acceptable lengths when you measure the cut board?

Relative error means the absolute value of the amount of error divided by the desired quantity. If x is the actual length after cutting, and the desired length is 13, then the amount of error is (13 – x) and the relative error would be: $\left|\frac{13 - x}{13}\right|$

Change the percent error to a decimal and set the relative error as less than or equal to .02: $\left|\frac{13 - x}{13}\right| \le 0.02$

Set up the compound inequality: $-0.02 \le \frac{13 - x}{13} \le 0.02$

Multiply all sections by 13: $-0.26 \le 13 - x \le 0.26$

Subtract 13 from all sections: $-13.26 \le -x \le -12.74$

Multiply all sections by negative 1: $13.26 \ge x \ge 12.74$

Put in the more usual order: $12.74 \le x \le 13.26$

To be acceptable, the cut board can measure anywhere on the interval [12.74, 13.26].

PROBLEMS

Solve:

1. $\left|\frac{3x-2}{5}\right| = 1$

2. $|n - 3| = |3 - n|$

3. $|5x| - 3 = 37$

4. $|x + 4| = |2x - 7|$

Solve and graph:

5. $40 - 4|a + 2| \geq 12$

6. $|2x - 1| + 7 > 18$

7. $\left|\frac{2-5x}{4}\right| \geq \frac{2}{3}$

8. $9 - |x + 4| \leq 5$

9. $|m + 5| + 9 < 26$

10. $|2y - 7| > -5$

ANSWERS

1. $\frac{3x-2}{5} = 1$ **OR** $\frac{3x-2}{5} = -1$

$3x - 2 = 5$ **OR** $3x - 2 = -5$

$3x = 7$ **OR** $3x = -3$

$x = 7/3$ **OR** $x = -1$

{-1, 7/3}

2. $|n - 3| = 3 - n$

$n - 3 = 3 - n$ **OR** $n - 3 = -3 + n$

$n = 3$ **OR** $n = n$ (always true)

$\mathbb{R}$ **or** **$(-\infty, \infty)$**

3. $|5x| = 40$

$5x = 40$ **OR** $5x = -40$

$x = 8$ **OR** $x = -8$

{-8, 8}

4. $|x + 4| = 2x - 7$

$x + 4 = 2x - 7$ **OR** $x + 4 = -2x + 7$

$x = 11$ **OR** $x = 1$

{1, 11}

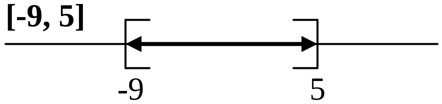
5. $-4|a + 2| \geq -28$

$|a + 2| \leq 7$ ← this determines **AND**

$a + 2 \leq 7$ **AND** $a + 2 \geq -7$

$a \leq 5$ **AND** $a \geq -9$

[-9, 5]



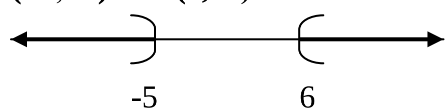
6. $|2x - 1| > 11$

$2x - 1 > 11$ **OR** $2x - 1 < -11$

$2x > 12$ **OR** $2x < -10$

$x > 6$ **OR** $x < -5$

$(-\infty, -5) \cup (6, \infty)$



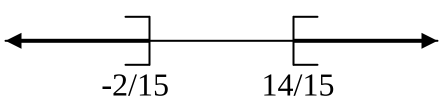
7. $\frac{2-5x}{4} \geq \frac{2}{3}$ **OR** $\frac{2-5x}{4} \leq -\frac{2}{3}$

$2 - 5x \geq \frac{8}{3}$ **OR** $2 - 5x \leq -\frac{8}{3}$

$-5x \geq \frac{2}{3}$ **OR** $-5x \leq -\frac{14}{3}$

$x \leq -\frac{2}{15}$ **OR** $x \geq \frac{14}{15}$

$(-\infty, -2/15] \cup [14/15, \infty)$



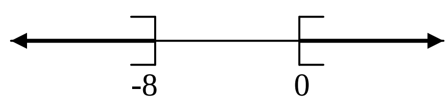
8. $-|x + 4| \leq -4$

$|x + 4| \geq 4$ ← this determines **OR**

$x + 4 \geq 4$ **OR** $x + 4 \leq -4$

$x \geq 0$ **OR** $x \leq -8$

$(-\infty, -8] \cup [0, \infty)$

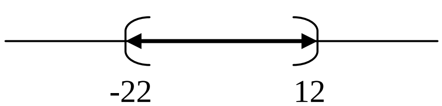


9. $|m + 5| < 17$

$m + 5 < 17$ **AND** $m + 5 > -17$

$m < 12$ **AND** $m > -22$

(-22, 12)



10. $\mathbb{R}$ **or** **$(-\infty, \infty)$**