

Theory of Polynomial Equations

A Quick Reference Sheet for Polynomials Having the General Form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$$

I) Eight Basic Theorems

1) Fundamental Theorem of Algebra: Every polynomial $P(x)$ of degree $N \geq 1$ has at least one solution x_0 for which $P(x_0) = 0$. This solution may be real or complex (i.e. has the form $a + bi$).

2) Numbers Theorem for Roots and Turning Points: If $P(x)$ is a polynomial of degree N , then the equation $P(x) = 0$ has up to N real solutions or roots. The equation $P(x) = 0$ has exactly N roots if one counts complex solutions of the form $a + bi$. Lastly, the graph of $P(x)$ will have up to $N - 1$ turning points (which includes both relative maxima and minima).

3) Real Root Theorem: If $P(x)$ is of odd degree having all real coefficients, then $P(x)$ has at least one real root.

4) Rational Root Theorem: If $P(x)$ has all integer coefficients, then any rational roots for the equation $P(x) = 0$ must have the form $\frac{p}{q}$ where p is a factor of the constant coefficient a_0 and q is a factor of the lead coefficient a_n . *Note: This result is used to form a rational-root possibility list.*

5) Complex Conjugate Pair Root Theorem: Suppose $P(x)$ has all real coefficients. If $a + bi$ is a root for $P(x)$ with $P(a + bi) = 0$, then $P(a - bi) = 0$.

6) Irrational Surd Pair Root Theorem: Suppose $P(x)$ has all rational coefficients. If $a + \sqrt{b}$ is a root for $P(x)$ with $P(a + \sqrt{b}) = 0$, then $P(a - \sqrt{b}) = 0$.

7) Remainder Theorem: If $P(x)$ is divided by $(x - c)$, then the remainder R is equal to $P(c)$. *Note: this result is extensively used to evaluate a given polynomial $P(x)$ at various values of x .*

8) Factor Theorem: If c is any number with $P(c) = 0$, then $(x - c)$ is a factor of $P(x)$. This means $P(x) = (x - c) \cdot Q(x)$ where $Q(x)$ is a new, reduced polynomial having degree one less than $P(x)$. The converse is also true $P(x) = (x - c) \cdot Q(x) \Rightarrow P(c) = 0$.

II) Three Advanced Theorems

9) Root Location Theorem: Let (a, b) be an interval on the x axis with $P(a) \cdot P(b) < 0$. Then there is a value $x_0 \in (a, b)$ such that $P(x_0) = 0$.

10) Root Bounding Theorem: Divide $P(x)$ by $(x - d)$ to obtain $P(x) = (x - d) \cdot Q(x) + R$. Case $d > 0$: If both R and all the coefficients of $Q(x)$ are positive, then $P(x)$ has no root $x_0 > d$. Case $d < 0$: If the roots of $Q(x)$ alternate in sign—with the remainder R "in sync" at the end—then $P(x)$ has no root $x_0 < d$. *Note: Coefficients of zero can be counted either as positive or negative—which ever way helps in the subsequent determination.*

11) Descartes' Rule of Signs: Arrange $P(x)$ in standard order as shown in the title bar. The number of positive real solutions equals the number of coefficient sign variations or that number decreased by an even number. Likewise, the number of negative real solutions equals the number of coefficient sign variations in $P(-x)$ or that number decreased by an even number.