13.3 Population Growth and Radioactive Decay

Assume $\#\text{ of births per year} = b \times p$, $\#\text{ of deaths per year} = d \times p$ where b is the annual birth rate and d is the annual death rate and p is the population.

The net increase in population $= \#\text{ of births} - \#\text{of deaths} = b \times p - d \times p = (b-d) \times p = r \times p$ where $r = b - d$ is the growth rate per year.

If $p_0 = \text{initial population}$

$$p_1 = p_0 + r \cdot p_0 = (1+r)^1 \cdot p_0$$
$$p_2 = p_1 + r \cdot p_1 = (1+r)^1 \cdot p_0 + r \cdot (1+r)^1 \cdot p_0 = (1+r)^1 \cdot p_0 \cdot (1+r) = (1+r)^2 \cdot p_0$$
$$p_3 = p_2 + r \cdot p_2 = (1+r)^2 \cdot p_0 + r \cdot (1+r)^2 \cdot p_0 = (1+r)^2 \cdot p_0 \cdot (1+r) = (1+r)^3 \cdot p_0$$
$$\vdots$$
$$p_m = p_{m-1} + r \cdot p_{m-1} = (1+r)^{m-1} \cdot p_0 + r \cdot (1+r)^{m-1} \cdot p_0 = (1+r)^{m-1} \cdot p_0 \cdot (1+r) = (1+r)^m \cdot p_0$$

Malthusian Population Growth

If the population is initially p_0, then after m years the population will be $p_m = (1+r)^m \cdot p_0$ where r is the growth rate per year.

Example: If the population of a small town is 2,000 and if it grows at a rate of 3% each year, then what is the approximate population in 10 years? 500 years?

Here $p = 2{,}000$ and $r = 3\% = .03$ and $m = 10$ so

$$p_{10} = (1+.03)^{10} \cdot 2000 = (1.03)^{10} \cdot 2000 = (1.3439) \cdot 2000 = 2687.8 \approx 2{,}700$$

Similarly,

$$p_{500} = (1+.03)^{500} \cdot 2000 = (1.03)^{500} \cdot 2000 = (2621877) \cdot 2000 = 5{,}243{,}754{,}468 \approx 5.2 \text{ billion}$$

Note that the predicted population after 500 years is very unrealistic. The Malthusian population model is valid only if the population growth remains constant. Over a large period of time that assumption is often not realistic.

If Q is the final population and P is the initial population then

$$Q = P \cdot (1+r)^m \Rightarrow (1+r)^m = \frac{Q}{P} \Rightarrow 1 + r = \left(\frac{Q}{P}\right)^{1/m} \Rightarrow r = \left(\frac{Q}{P}\right)^{1/m} - 1$$

Example: Suppose an initial population of 2,000 grows to 2,500 over 15 years. What is the growth rate? What is the predicted population 25 years after the population was 2,000?

$$r = \left(\frac{2{,}500}{2{,}000}\right)^{\frac{1}{15}} - 1 = (1.25)^{\frac{1}{15}} - 1 = 1.015 - 1 = .015 = 1.5\%$$

$$p_{25} = 2{,}000 \cdot (1 + .015)^{25} = 2901.9 \approx 2{,}900$$

<u>Radioactive Decay</u>

If a radioactive substance has an annual decay rate of d and if there is initially A_0 units of the substance present, then after m years there will be $(1-d)^m \cdot A_0$ units present.

<u>Definition:</u> The half-life of a substance is the time after which half of the substance is present.

After a half-life of h years,

$$\frac{1}{2}A_0 = (1-d)^h \cdot A_0 \Rightarrow$$

$$(1-d)^h = \frac{1}{2} \Rightarrow$$

$$1 - d = \left(\frac{1}{2}\right)^{1/h} \Rightarrow$$

$$d = 1 - \left(\frac{1}{2}\right)^{1/h}$$

Now if you plug in $d = 1 - \left(\frac{1}{2}\right)^{1/h}$ into $A_m = (1-d)^m \cdot A_0$ you will get

$$A_m = \left(1 - \left(1 - \left(\frac{1}{2}\right)^{1/h}\right)\right)^m \cdot A_0 = \left(\left(\frac{1}{2}\right)^{1/h}\right)^m \cdot A_0 = \left(\frac{1}{2}\right)^{\frac{m}{h}} \cdot A_0$$

In summary, there are two formulas you can use to find the amount of substance remaining after m years.

$A_m = (1-d)^m \cdot A_0$ which works well if you know the decay rate

$A_m = \left(\frac{1}{2}\right)^{\frac{m}{h}} \cdot A_0$ which works well if you know the half life

Example: Suppose the half-life of a radioactive substance is 8.5 years. What is the annual decay rate? If you have 100 grams in 1995, how much is left in 2025?

$$d = 1 - \left(\frac{1}{2}\right)^{\frac{1}{8.5}} = 1 - (.5)^{.11765} = 1 - .9217 = .0783 = 7.8\%$$

To find the amount remaining in 2025, you can use either formula.
Here $h = 8.5, m = 30, A_0 = 100$, so

$$A_{30} = \left(\frac{1}{2}\right)^{\frac{30}{8.5}} \cdot 100 = (.5)^{3.5294} \cdot 100 = .0866 \cdot 100 \approx 8.7 \text{ grams}$$

or

$$A_{30} = (1 - .0783)^{30} \cdot 100 = (.9217)^{30} \cdot 100 = .0866 \cdot 100 \approx 8.7 \text{ grams}$$

Carbon Dating

^{14}C produced by cosmic radiation (via neutrons) enters a living object via CO_2 and at death the amount of ^{14}C is at its maximum and decays according to:

$A_m = \left(\frac{1}{2}\right)^{\frac{m}{h}} \cdot A_0$ where m is the age of the substance, $h = 5{,}600$ yrs is the half-life of ^{14}C and A_0 is the amount of ^{14}C present at death. So the % of ^{14}C present m years after death is $\frac{A}{A_0} = \left(\frac{1}{2}\right)^{\frac{m}{h}}$

Logistic Population Model

The maximum population size the environment can support is called the carrying capacity c.

$p_{m+1} = (1 + r) \cdot p_m - \left(\frac{1+r}{c}\right) \cdot p_m^2$ where m is the number of breeding seasons and r is the natural growth rate (assuming a very large container), and c is the carrying capacity. The above equation is called the logistic law, or logistic equation. Unlike the previous equations the logistic equation is iterative. You have to calculate one breeding season at a time until you work your way up to the season you are interested in.

Example: Suppose a population is governed by the logistic law with $c = 2{,}000$ and $r = 2.7$. If the initial population size is 1,000 find the population after 3 breeding seasons.

$$p_0 = 1{,}000$$
$$p_1 = (1+2.7)\cdot 1{,}000 - \left(\frac{1+2.7}{2{,}000}\right)\cdot 1{,}000^2 = 3{,}700 - 1{,}850 = 1{,}850$$
$$p_2 = (1+2.7)\cdot 1{,}850 - \left(\frac{1+2.7}{2{,}000}\right)\cdot 1{,}850^2 = 6{,}845 - 6332 = 513$$
$$p_3 = (1+2.7)\cdot 513 - \left(\frac{1+2.7}{2{,}000}\right)\cdot 513^2 = 1{,}898 - 486 = 1{,}412$$

<u>Stable Population under the Logistic Law</u>

If a population has a natural growth rate of r and the environment has a carrying capacity of c, then the stable population size is $\frac{r \times c}{1+r}$. Once a population reaches the stable population size it will remain fixed.

Example: For the problem above $r = 2.7, c = 2{,}000$ so the stable population size is $\frac{2.7 \times 2{,}000}{1+2.7} \approx 1{,}459$

The notes above are for Math 108, Math for the Modern World using *Mathematics in Life, Society and the World 2nd edition by Parks, Musser, Burton, and Siebler. Prentice Hall 2000.*