10.1 Statements and Logical Connectives

Definition: A statement is a sentence that can be classified as either true or false. (Statements are often represented by lowercase letters)

Example: "The moon is not a star"

Example: "2 + 2 = 10"

Example: "Math is the best major" is NOT a statement because it is subjective

Example: "The house" is NOT a statement because it is not a sentence

Definition: The negation of a statement p, written ~p (read not p), is the "opposite" of the statement.

Example: The negation of "The sun is bigger than the earth" is "The sun is not bigger than the earth"

Truth Table:

p	~p
T	F
F	T

Logical Connectives

Definition: The conjunction of p and q, written $p \wedge q$, is true only when both p and q are true

Note: The conjunction (AND) is *strict*, meaning it is only true when both p and q are true

Truth Table:

p	q	$p \wedge q$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**F**

Example: "Houses are bigger than atoms and 2 + 2 = 5" is false

Definition: The disjunction of p and q, written $p \vee q$, is true if either p or q is true

Note: The disjunction (OR) is *lenient*, meaning it is only true when either p or q or both are true

Truth Table:

p	q	$p \vee q$
T	**T**	**T**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

Example: "Houses are bigger than atoms and 2 + 2 = 5" is true

Example: If p represents "Fire is hotter than ice" and q represents "Cats are the same kind of animals as dogs", then which of the following is true?

A) $\sim q$

Since q is false, ~q is TRUE

B) $p \vee q$

Since p is true, it is TRUE

C) $p \wedge q$

Since p and q are not both true, it is FALSE

D) $(\sim p) \vee q$

Since ~p is false and q is false, it is FALSE

E) $\sim (p \wedge q)$

Since part C) above is false, it is TRUE

<u>If-Then</u>

Definition: The statement "If p then q", written $p \Rightarrow q$ is called an <u>implication</u> or <u>conditional statement</u> (p is called the <u>antecedent</u> or <u>hypothesis</u> and q is called the <u>consequent</u> or <u>conclusion</u>)

Truth Table:

p	q	$p \Rightarrow q$
T	**T**	**T**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

To understand the table you may want to think of $p \Rightarrow q$ as a promise (If you do p then I will do q). The promise should be considered kept (TRUE) unless there is clear evidence that it was broken. In the second row of the table you did p, but I did not do q, so the promise was broken. In the bottom two rows, you did not do p, so the promise was not relevant, therefore I did not break my word so it is TRUE.

Example: Which of the following conditionals are true?

A) "If mars is a planet then 2 + 2 = 4"
p and q are both true, so it is TRUE

B) "If 3 is divisible by 2 then 3 is even"
p is false, so it is TRUE

C) "If the sun is a star then pigs can fly"
p is true and q is false so it is FALSE

Related Conditionals

Definition: The converse of "If p then q" is "If q then p",
i.e. the converse of $p \Rightarrow q$ is $q \Rightarrow p$

Example: The converse of
"If an integer greater than 2 is prime then it is not even" is
"If an integer greater than 2 is not even then it is prime"

Note: The converse may not be true even if the original conditional was true.
The converse is NOT logically equivalent to the original conditional.

Definition: The contrapositive of "If p then q" is "If not q then not p",
i.e. the contrapositive of $p \Rightarrow q$ is $\sim q \Rightarrow \sim p$

Example: The contrapositive of
"If an integer greater than 2 is prime then it is not even" is
"If an integer greater than 2 is even then it is not prime"

Note: The contrapositive will be true whenever the original conditional is true.
The contrapositive IS logically equivalent to the original conditional.

Definition: The inverse of "If p then q" is "If not p then not q",
i.e. the inverse of $p \Rightarrow q$ is $\sim p \Rightarrow \sim q$

Example: The inverse of
"If an integer greater than 2 is prime then it is not even" is
"If an integer greater than 2 is not prime then it is even"

Note: The inverse may not be true even if the original conditional was true.
The inverse is NOT logically equivalent to the original conditional.

Example: For the statement: "If it is snowing then you will have your coat on", identify the hypothesis and conclusion. State the converse, inverse, and contrapositive.

Hypothesis: "If it is snowing"
Conclusion: "you will have your coat on"

Converse: "If you have your coat on then it is snowing"
Inverse: "If it is not snowing then you will not have your coat on"
Contrapositive: "If you do not have your coat on then it is not snowing"

Note:

p	q	$\sim p$	$\sim q$	$p \Rightarrow q$	$\sim q \Rightarrow \sim p$	$q \Rightarrow p$	$\sim p \Rightarrow \sim q$
T	T	F	F	T	T	T	T
T	F	F	T	F	F	T	T
F	T	T	F	T	T	F	F
F	F	T	T	T	T	T	T

Identical, so logically equivalent   Identical, so logically equivalent

This proves that the original conditional and its contrapositive are logically equivalent. The converse and inverse are not logically equivalent to the original conditional, but they are logically equivalent to each other.

If and only if

Definition: The biconditional "p if and only if q", written $p \Leftrightarrow q$ is true when both p and q are true, or p and q are false.

Truth Table:

p	q	$p \Leftrightarrow q$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**T**

Example: Which of the following biconditionals are true?

A) "Ice is cold if and only if fire is hot"
TRUE

B) "2 is even if and only if $2 + 2 = 6$"
FALSE

C) "$3 + 3 = 8$ if and only if red is blue"
TRUE

Note:

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$(p \Rightarrow q) \wedge (q \Rightarrow p)$	$p \Leftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

Identical, so logically equivalent

So $p \Leftrightarrow q$ can be thought of as meaning $p \Rightarrow q$ AND $q \Rightarrow p$

The notes above are for Math 108, Math for the Modern World using

Mathematics in Life, Society and the World 2^{nd} *edition by Parks, Musser, Burton, and Siebler. Prentice Hall 2000.*